Two hours

UNIVERSITY OF MANCHESTER
SCHOOL OF COMPUTER SCIENCE

Mathematical Techniques for Computer Science

Time:

Please answer all THREE Questions

Use a SEPARATE Answerbook for each SECTION

This is a CLOSED book examination

The use of electronic calculators is permitted provided they are not programmable and do not store text or formulas.

Section A

1. a) Show by induction that for all $n \in \mathbb{N} \setminus \{0\}$ we have (4 marks)

$$\sum_{i=1}^{n} 3i - 2 = \frac{n(3n-1)}{2}.$$

b) Consider the set $\mathsf{Lists}_{\mathbb{N}}$ of lists with entries from $\mathbb{N}$ from the notes.

i) Recursively define a function

$$\text{dltodd}\colon \mathsf{Lists}_{\mathbb{N}} \longrightarrow \mathsf{Lists}_{\mathbb{N}},$$

which deletes all the odd numbers from the given list, so that we have, for example, (3 marks)

$$\text{dltodd}[3,2,1,0] = [2,0].$$

ii) In the notes there is a Java class List which is the direct counterpart of a recursively defined list. For this class give a method that implements the function described in part i) by filling in the return cases in the following Java program. (3 marks)

```
public static List ins1 (List l)
{
if (l == null)
    return
else



}
```

iii) Recall the following operation on lists from the notes:

Base case sum. $\text{sum}\,[\,] = 0.$

Step case sum. $\text{sum}(s : l) = s + \text{sum}\, l$

Show that for all $l \in \mathsf{Lists}_{\mathbb{N}}$ we have that (4 marks)

$$\text{sum}\,\text{dltodd}\, l \qquad \text{is even.}$$

c) Assume that we have a propositional formula A_i for each $i \in \mathbb{N} \setminus \{0\}$. Using the De Morgan law for two formulae,

$$\neg(A_1 \vee A_2) \equiv \neg A_1 \wedge \neg A_2,$$

show by induction that for all $n \in \mathbb{N} \setminus \{0, 1\}$ we have that (4 marks)

$$\neg(A_1 \vee A_2 \vee \cdots \vee A_n) \equiv \neg A_1 \wedge \neg A_2 \wedge \cdots \wedge A_n.$$

d) What does it mean for a set to be countably infinite? Make sure you use definitions or results from the notes in your answer. (2 marks)

2. a) Check whether the relations defined below are equivalence relations. Justify your answers. (6 marks)

   i) On matrices over the set of real numbers: The matrix $\underline{A}$ is related to the matrix $\underline{B}$ if and only if for every column vector $\underline{b}$ in $\underline{B}$ there is a column vector $\underline{a}$ in $\underline{A}$ and a real number α with the property that

   $$\alpha\underline{a} = \underline{b}.$$

   ii) On propositional formulae: We define that the propositional formula A is related to the propositional formula B if and only if A and B contain the same propositional variables and if for every boolean valuation v for those propositional variables it is the case that

   $$vA = vB.$$

   b) Check whether the following relations are partial orders. Justify your answers. For any partial order you identify sketch a Hasse diagram with at least six elements. (7 marks)

   i) There is a league of teams, and each team plays every other time exactly once in the course of a season. At the end of the season we say that Team A is related to Team B if Team A was beaten by Team B.

   ii) Let P be the set of prime numbers, starting with 2, 3, 5, 7. Assume we have lists l and l' in Lists_P whose entries are ordered, that is, the elements have been added in rising order. We say that l is related to l' if and only if the product of all elements in l is less than or equal to that of all elements of l'.

c) Assume you have been asked to implement a number of Java classes for geometric figures and you have been asked to give a diagram that illustrates which of these classes are subclasses of each other. You have been asked to ensure that your subclass relation reflects the usual 'is an instance of' relation (so a square is an instance of a rectangle). The classes you are to implement are the following.

- Figure
- Circle
- Rectangle
- Square
- Polygon
- Parallelogram
- Rhombus

We know from the notes that the 'is a subclass of' relation is a partial order.

(4 marks)

i) Draw a Hasse diagram for this relation for these classes.
ii) What are the minimal elements of this partial order? Is there a least element?
iii) What are the maximal elements of this partial order? Is there a greatest element?
iv) Is there a least upper bound for Square and Rhombus? If yes what is it?

d) Consider the following equivalence relation on integers: For m and n we have

$$m \sim n \qquad \text{iff} \qquad m = n \mod 6.$$

Consider the following function:

$$\begin{aligned} f\colon \mathbb{Z} &\longrightarrow \mathbb{Z} \\ n &\longmapsto n \bmod 3. \end{aligned}$$

Show that the function f is constant on equivalence classes, that is, two elements which are equivalent are mapped to the same element. (3 marks)

Section B

3. a) Let $\underline{a} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\underline{b} = \begin{pmatrix} -4 \\ -\pi \\ 2 \end{pmatrix}$.

i) Find a vector $\underline{v}$ such that (2 marks)

$$\underline{a} + \underline{b} + \underline{v} = \underline{0}.$$

ii) Give a geometric interpretation for this task, i.e., say what it means to find such a vector $\underline{v}$ so that the equation holds. One or two sentences should suffice. (1 mark)

b) We say that a matrix $\underline{A}$ commutes with a matrix $\underline{B}$, if $\underline{AB} = \underline{BA}$.

i) Find all values of k for which (4 marks)

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & k \end{pmatrix} \quad \text{commutes with} \quad \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

ii) Show that matrix multiplication is not commutative. (1 mark)

c) In each of the following, is the given function a linear transformation? Say why, and where possible give its matrix representation. (4 marks)

i) $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ defined by $T \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_1 + 2x_2 \\ x_2 \end{pmatrix}$

ii) $S : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ defined by $S \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_1 + 2 \end{pmatrix}$

d) Let $\underline{A}$ and $\underline{B}$ be square matrices such that $\underline{AB} = \underline{A}$ and $\underline{B}^2 = \underline{0}$. Show that:

$$\underline{A} = \underline{0}.$$

(2 marks)

e) Consider $\underline{Ax} = \underline{b}$, where

$$\underline{A} = \begin{pmatrix} -1 & 1 & 4 \\ 1 & 3 & 8 \\ \frac{1}{2} & 1 & \frac{5}{2} \end{pmatrix} \quad \text{and} \quad \underline{b} = \begin{pmatrix} 12 \\ 8 \\ 2 \end{pmatrix}.$$

i) Write down a system of linear equations for $\underline{Ax} = \underline{b}$. (1 mark)

ii) Use our Gaussian elimination method to determine if $\underline{Ax} = \underline{b}$ has infinitely many solutions, has a unique solution or has no solution. (3 marks)

iii) Give the solution set of system $\underline{Ax} = \underline{b}$.
Give the solution set for the homogeneous system $\underline{Ax} = \underline{0}$. (2 marks)

END OF EXAMINATION