COMP36111: Advanced Algorithms I

Lecture 4: Just married!

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In this lecture, we continue the theme of matching, this time supposing that the matched individuals have preferences expressed as rankings. We define a notion of ‘stability’ for such matchings, and show how to compute a stable matching in polynomial time. The lecture has two parts:

• the stable marriage problem;
• the Gale Shapley algorithm.
Outline

The stable marriage problem

The Gale-Shapley algorithm

Summary
The stable marriage problem

The Gale-Shapley algorithm

Summary

Humphrey Bogart
Lauren Bacall
Al Roth
Lloyd Shapley
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The stable marriage problem

The Gale-Shapley algorithm

Summary

Humphrey Bogart  Lauren Bacall

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The stable marriage problem

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Consider the following (slightly idealized) matchmaking problem.

We are given:

- a set of $n$ boys and $n$ girls;
- a strict ranking, for each boy, of all the girls
- a strict ranking, for each girl, of all the boys

We want to compute:

- a 1–1 pairing of boys with girls in which, for every boy $a$ and girl $b$, either $a$ prefers his partner to $b$ or $b$ prefers her partner to $a$. (Such a pairing is called a stable matching.)
• It helps to draw the arrangement as a complete bipartite graph:

![Graph](image)

- boys
- girls

• It is not in the least obvious that a stable matching always exists.
• It helps to draw the arrangement as a complete bipartite graph:

```
   3   5
   2   4
   1
```

boys    girls

• It is not in the least obvious that a stable matching always exists.
Outline

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Summary
• The Gale-Shapley algorithm generates a matching as follows

\begin{verbatim}
begin Gale-Shapley(Boys' rankings, Girls' rankings)
    until all boys are engaged do
        for each boy $a$ with no fiancée do
            $a$ proposes to the girl he most prefers among
            those that $a$ has not yet proposed to
        for each girl $b$ with new proposals do
            let $a$ be $b$'s most preferred new suitor
            if $b$ has no fiancé
                $b$ gets engaged to $a$
            if $b$ prefers $a$ to her existing fiancé
                $b$ cancels her existing engagement
                $b$ gets engaged to $a$
    All the engaged couples get married
the end
\end{verbatim}
Theorem

The Gale-Shapley algorithm terminates with a stable matching.

Proof.
The algorithm terminates, since no boy proposes to any girl twice.
Theorem

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When the algorithm terminates, every girl will have received a proposal, since any unmatched boys will have proposed to all the girls.
Theorem

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When the algorithm terminates, every girl will have received a proposal, since any unmatched boys will have proposed to all the girls.

Once a girl has received a proposal, she always has some fiancé or other. Hence all the girls get engaged. So we have a matching.
Theorem

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Suppose $\langle a, b \rangle$ and $\langle a', b' \rangle$, are distinct married couples produced by the algorithm. If $b$ prefers $a'$ to $a$, then $a'$ never proposed to $b$. But $a'$ proposed to all girls better (for him) than or equal to $b'$. Therefore, $a'$ does not prefer $b$ to $b'$.\[\]
Exercise: show that Gale-Shapley terminates in time $O(n^2)$, where $n$ is the number of boys (or girls).
Theorem
The stable matching, $M$, produced by the Gale-Shapley algorithm is optimal for boys: if boy $a$ is married to girl $b$ in $M$, but prefers girl $b'$, then there is no stable matching $M'$ in which $a$ is married to $b'$.

Proof.

If $a$ prefers $b'$ to $b$, then the pair $\langle a, b' \rangle \in M'$ must have been a rejected proposal or cancelled engagement in the construction of $M$. 
The stable marriage problem

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Theorem

The stable matching, \( M \), produced by the Gale-Shapley algorithm is optimal for boys: if boy \( a \) is married to girl \( b \) in \( M \), but prefers girl \( b' \), then there is no stable matching \( M' \) in which \( a \) is married to \( b' \).

Proof.

If \( a \) prefers \( b' \) to \( b \), then the pair \( \langle a, b' \rangle \in M' \) must have been a rejected proposal or cancelled engagement in the construction of \( M \).
**Proof.**

Let \( \langle a, b' \rangle \in M' \) be the first pair rejected or cancelled in the construction of \( M \).

![Gale-Shapley diagram](image)

![Stable matching \( M' \)](image)

Let \( a'' \) be such that \( b' \) rejects/cancels \( a \) for \( a'' \) in construction of \( M \). So \( b' \) prefers \( a'' \) to \( a \).

Let \( b'' \) be the partner of \( a'' \) in \( M' \) (stable). So \( a'' \) prefers \( b'' \) to \( b' \).

But then the pair \( \langle a'', b'' \rangle \) must have been rejected/cancelled in construction of \( M \), contradicting fact that \( \langle a, b' \rangle \) is first. \( \square \)
Proof.
Let $\langle a, b' \rangle \in M'$ be the first pair rejected or cancelled in the construction of $M$.

Let $a''$ be such that $b'$ rejects/cancels $a$ for $a''$ in construction of $M$. So $b'$ prefers $a''$ to $a$.

Let $b''$ be the partner of $a''$ in $M'$ (stable). So $a''$ prefers $b''$ to $b'$.

But then the pair $\langle a'', b'' \rangle$ must have been rejected/cancelled in construction of $M$, contradicting fact that $\langle a, b' \rangle$ is first. \qed
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- In this lecture, we have considered:
  - the Gale-Shapley algorithm for computing stable matchings in ranked bipartite graphs.
- Reading: