COMP36111: Advanced Algorithms I
Lecture 10: For Completeness’ Sake!

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QBF is $\text{PSPACE}$-complete

REACHABILITY is $\text{NLOGSPACE}$-complete

Conclusion
Theorem

The problem QBF is \( PSPACE \)-complete.

Proof.

We have already shown (in Lecture 7) that QBF is in \( PSPACE \).

For \( PSPACE \)-completeness, suppose \( L \) is in \( PSPACE \), and let \( M \) be a deterministic Turing Machine recognizing \( L \) and running with space bound \( f(n) \), where \( f \) is a polynomial. Fix an input \( x \) for \( M \) of length \( n \), and let \( G \) be the configuration graph of \( L \) with input \( x \). The number of vertices in \( G \) is given by \( c^{f(n)} \) where \( c \) is a constant depending only on \( M \).

Let \( u_0 \) be the initial configuration with input \( x \) and \( v_0 \) the successful halting configuration. We want to know whether there is a path from \( u_0 \) to \( v_0 \) of length at most \( c^{f(n)} \). \(\square\)
Contd.
We encode each configuration with a sequence $\bar{p}$ of proposition letters of length $f(n) + 2$, and we write a formulas of propositional logic $\psi_0(\bar{p}, \bar{q})$ stating that $M$ can transition from the configuration encoded by $\bar{p}$ to that encoded by $\bar{q}$. This is routine, and depends only on $M$.

It suffices to write, for each $i$ ($1 \leq i \leq \log c^{f(n)} = f(n) \cdot \log c$), a QB-formula $\psi_i(\bar{p}, \bar{q})$ true if and only $\bar{q}$ is reachable from $\bar{p}$ in at most $2^i$ steps. Note that $\psi_0$ has already been defined.

Assume that $\psi_i$ has been defined. Our first thought might be to write

$$\psi_{i+1}(\bar{p}, \bar{q}) := \exists \bar{r}(\psi_i(\bar{p}, \bar{r}) \land \psi_i(\bar{r}, \bar{q})).$$
Contd.

But this would mean that $\psi_i$ doubles in size when we increment $i$; we need to construct all the $\psi_i$ in polynomial time (actually, in logarithmic space).

So, instead and write

$$\psi_{i+1}(\bar{p}, \bar{q}) := \exists \bar{r} \forall \bar{a} \forall \bar{b}(((\bar{a} = \bar{p} \land \bar{b} = \bar{r}) \lor (\bar{a} = \bar{r} \land \bar{b} = \bar{q})) \rightarrow \psi_i(\bar{a}, \bar{b})).$$

This does the trick. It is then easy to map $x$ to a QB-formula $\varphi_x$ such that $x$ is in accepted by $M$ (i.e. $x \in L$) if and only if $\varphi_x$ is true.

N.b. $\bar{a} = \bar{p}$ abbreviates $(a_1 \leftrightarrow p_1) \land (a_2 \leftrightarrow p_2) \land \cdots$. 
Outline

QBF is $\text{PSPACE}$-complete

REACHABILITY is $\text{NLOGSPACE}$-complete

Conclusion
• Recall the problem of reachability in directed graphs:

**REACHABILITY**
Given: A directed graph $G = (V, E)$ and nodes $s, t \in V$
Return: Yes if $t$ is reachable from $s$ in $G$, No otherwise.
Theorem (Revision!)

*REACHABILITY* is in *NLogSpace*.

**Proof.**

Here is an obvious non-deterministic procedure:

begin reachND(G, u, v)
    n := number of vertices in G
    w := u
    c := 0
    while w ≠ v and c < n
        pick w' such that w = w' or there is an edge from w to w'
        w := w'
        increment c;
    if c < n
        return Y
    return N
Theorem

*REACHABILITY is $\text{NLogSpace}$-complete.*

Proof.

We showed above that REACHABILITY is in $\text{NLogSpace}$.

Suppose $L$ is a language recognized by a non-deterministic TM, $M$, running in space $O(\log n)$. Given an input $x$, let $G$ be the configuration graph for $M$ with input $x$. Let $u$ be the node representing the initial configuration. We may assume this graph has a single accepting node $v$. Now, $x \in L$ if and only if $(G, u, v)$ is an instance of REACHABILITY. The mapping $x \mapsto (G, u, v)$ can easily be constructed in space bounded by $\log n$. •
Outline

QBF is $PSPACE$-complete

REACHABILITY is $NLOGSPACE$-complete

Conclusion
• In this lecture, we enlarged our repertoire of completeness results:
  • QBF-Sat is \( \text{PSPACE} \)-complete;
  • REACHABILITY is \( \text{NLOGSPACE} \)-complete.
• Reading for this lecture:
  • Sipser, Ch. 8 (Space Complexity). You need not read “Winning strategies for games” or “Generalized geography”.