1. To show membership in $\text{NP}$-time, let an instance $(U, x, y)$ of $3\text{D-MATCH}$ be given. Guess a set $T$ of $n = |U| = |x| = |y|$ triples $(u, x, y)$ and check that each element of $U \cup x \cup y$ occurs in exactly one triple. This check can clearly be performed in polynomial time, and thus yields a nondeterministic polynomial time procedure for $3\text{D-MATCH}$.

2. Write $a_{ij} = a_{i+1}$. Then $T_i = \sum_{i=1}^{m} t_{ij}$, $\overline{T_i} = \sum_{i=1}^{m} \overline{t_{ij}}$.

Then $Z^+ = \sum_{i=1}^{m} T_i \text{ satisfies (i)}$

$Z^- = \sum_{i=1}^{m} \overline{T_i} \text{ satisfies (ii)}$
For (iii), suppose \( Z \) is such that each \( a_{ij} \) and each \( b_{ij} \) occur in exactly one \( t \in Z \).

Let \( t_i \) be the triple of \( Z \) containing \( a_{ii} \). Then \( t_i \) is either \( t_{ii} \) or \( t_{in} \). If the former, \( t_{ii} \in Z \) and hence \( b_{iz} \in Z \) and hence \( t_{iz} \in Z \) and so on, whence \( Z = \{ \overline{t}_{ii}, \ldots, \overline{t}_{im} \} \). If the latter \( t_i = \{ \overline{t}_{ii}, \ldots, \overline{t}_{im} \} \) by an identical argument.

3. Suppose \( Z \) has the advertised properties. We construct a satisfying assignment \( \Theta \) for \( \Pi \).

Fix \( i \) \((1 \leq i \leq n)\). Let \( Z_i = Z \cap T_i \), i.e. the set of triples of \( Z \) involving the element \( a_{ij} \) or \( b_{ij} \). Thus, \( Z_i \) satisfies the conditions of \( \Pi Z(iii) \) and so is either \( Z_i^+ \) or \( Z_i^- \).

Let \( \Theta(p_i) = \{ T \text{ if } Z_i = Z_i^+ \} \)

This defines \( \Theta \). We claim \( \Theta \) is a satisfying assignment for \( \Pi \). Fix \( j \) \((1 \leq j \leq m)\). Then \( c_{j}d_{j} \) must be in one of the triples of \( Z \) and this can only be either \( \{ a_{ij}, c_{j}, d_{j} \} \) or \( \{ b_{ij}, c_{j}, d_{j} \} \).

If the former, \( Z_i \) cannot contain \( a_{ij} \) (since \( \{ a_{ij}, c_{j}, d_{j} \} \in Z \) hence \( Z_i = Z_i^- \) whence \( \Theta(p_i) = T \), whence \( \Theta(s_{ij}) = T \) (since \( p_i \) occurs in \( s_{ij} \)).

If the latter, \( Z_i = Z_i^+ \) by a symmetric argument, and \( \Theta(s_{ij}) = T \) (since \( p_i \) occurs in \( s_{ij} \)).
Some each \(c_{i,j}\) occur only in \(S_{j}\).

\( \forall S_{j}\) is a singleton and the triple it contains contains exactly one element \(u_{i,j}\) or \(\overline{u}_{i,j}\) for some \(i\). Thus, \(\nu_{i,j}\) total (over all \(j\)).

Each \(\forall S_{j}\) contains exactly \(m\) triples (by Q2), accounting for exactly \(m\) objects \(u_{i,j}\) or \(\overline{u}_{i,j}\) for \(j\). Thus: \(\mu_{i,j}\) total (over all \(j\)).

The total number of elements \(u_{i,j}\) or \(\overline{u}_{i,j}\) accounted for by \(Z\) is thus \(n.m + m\). Hence the number unaccounted for is

\[
2mn - (n.m + m) = m(n-1).
\]

4. Let \(R = \{ (u_{i,j}, g_{k}, h_{k}) \mid 1 \leq i \leq n, 1 \leq j \leq m, 1 \leq k \leq m(n-1) \} \cup \{ (\overline{u}_{i,j}, g_{k}, h_{k}) \mid 1 \leq i \leq n, 1 \leq j \leq m, 1 \leq k \leq m(n-1) \}\). (Think of \(R\) as a "rubbish dump") Suppose \(\Theta\) is a satisfying assignment for \(R\).

Let \(Z' = \{ (u_{i,j}, a_{i,j}, b_{i,j}) \mid 1 \leq i \leq n, 1 \leq j \leq m, \Theta(p_{i}) = F \} \cup \{ (\overline{u}_{i,j}, a_{i,j}, b_{i,j}) \mid 1 \leq i \leq n, 1 \leq j \leq m, \Theta(p_{i}) = T \} \).

Now, for each \(j\) (\(1 \leq j \leq m\)) pick some literal \(L_{j}\) from \(S_{j}\) such that \(\Theta(L_{j}) = T\), and let

\[
Z'' = \{ (u_{i,j}, c_{i,j}, d_{i}) \mid 1 \leq i \leq n, 1 \leq j \leq m, L_{j} = p_{i} \} \cup \{ (\overline{u}_{i,j}, c_{i,j}, d_{i}) \mid 1 \leq i \leq n, 1 \leq j \leq m, L_{j} = \overline{p}_{i} \}.
\]
This basis exactly $m(n-1)$ objects $u_{i,j}$ unaccounted for. For each such $u_{i,j}$ (not contained in any triple of $Z'$ or of $Z''$) define a new triple $r_k = (u_{i,j}, b_k, h_k)$, with $k$ running from 1 to $m(n-1)$. Let
\[ Z'' = \{ r_1, \ldots, r_{m(n-1)} \}. \]

Then $Z = Z' \cup Z'' \cup Z'''$ has the desired property.

5. Membership of 3D-MATCH in PTime is QL.
For NP-hardness, we proceed by reduction from SAT. Let an instance of SAT (collection of clauses) $\Gamma$ be given over propositional letters $p_1, \ldots, p_n$. Write $\Gamma = \{ \psi_1, \ldots, \psi_m \}$. Define two sets
\[ U_\psi = U_1 \cup \cdots \cup U_m, \]
\[ X_\psi = \{ (a_{i,j}, c_{i,j}, b_k) \mid 1 \leq i \leq n, 1 \leq j \leq n, 1 \leq k \leq m(n-1) \}, \]
\[ Y_\psi = \{ (b_{i,j}, d_{i,j}, h_k) \mid 1 \leq i \leq n, 1 \leq j \leq n, 1 \leq k \leq m(n-1) \}. \]

where the $U_i, a_{i,j}, b_k, c_{i,j}, d_{i,j}, g_k, h_k$ are as in question 2-4. Thus, $(U_\psi, X_\psi, Y_\psi)$ is an instance of 3D-MATCH, and can evidently be constructed using only logarithmic space. We claim $\Gamma$ is satisfiable if and only if $(U_\psi, X_\psi, Y_\psi)$ is a positive instance of 3D-MATCH. For the $\Rightarrow$-direction, let $Z^*$ be a witnessing set of triples. Then $Z = Z^* \setminus R$ satisfying the conditions of Q3 and so $\Gamma$ is satisfiable. The only-$\Leftarrow$ direction is Q4.