Lecture 9
Algorithmic Techniques Part 3B

COMP26120

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Recap

Linear Programming problems are of the form

Maximize:	$\mathbf{c}^T\mathbf{x}$
Subject to:	$\mathbf{A}\mathbf{x} \leq \mathbf{b}$
	$\mathbf{x} \geq 0$

$\mathbf{c}^T\mathbf{x}$ is the objective function and $\mathbf{A}\mathbf{x} \leq \mathbf{b}$ are the constraints

To get into this standard form we may need to expand equalities, reverse $\geq$, enforce ≥ 0, or swap coefficients to go from min to max.

Linear programming has real solutions and is in P. Integer Linear Programming (ILP) has integer solutions and is NP-complete.

Thinking Geometrically

The constraints $\mathbf{Ax} \leq \mathbf{b}$ form a feasible region that is a convex polytope.

For any linear objective function the optima only occur at the corners (vertices) of the feasible region (never inside).

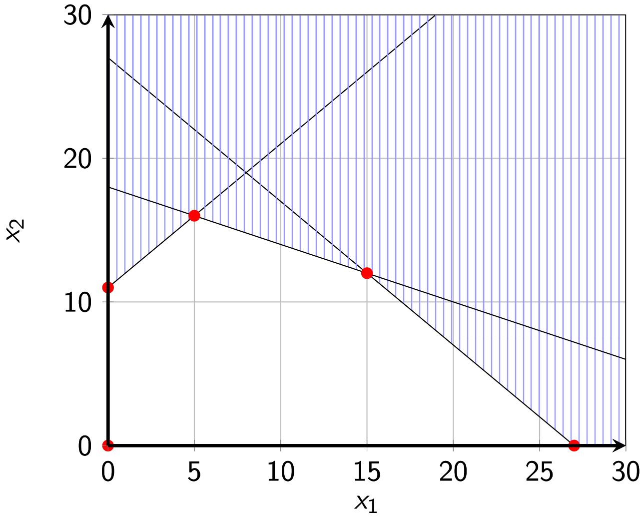
Vertices are given by intersections of constraints.

An intersection does not need to be a point (there may not be a unique optimum).

A solution at an intersection is a basic solution, if it is in the feasible region it is a basic feasible solutions

A Simple Convex Polygon

$$-x_1 + x_2 \leq 11$$
$$x_1 + x_2 \leq 27$$
$$2x_1 + 5x_2 \leq 90$$
$$x_1, x_2 \geq 0$$

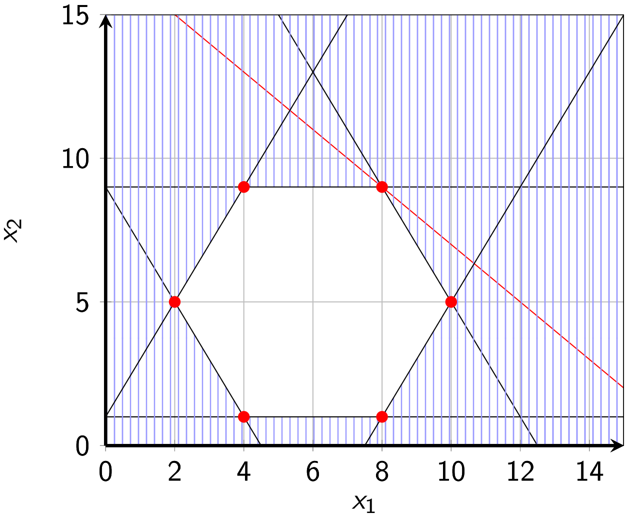


A Less Simple Convex Polygon

$$x_2 \leq 9$$
$$-x_2 \leq -1$$
$$2x_1 + x_2 \leq 25$$
$$-2x_1 - x_2 \leq -9$$
$$-2x_1 + x_2 \leq 1$$
$$2x_1 - x_2 \leq 15$$

Maximize
$z = x_1 + x_2$

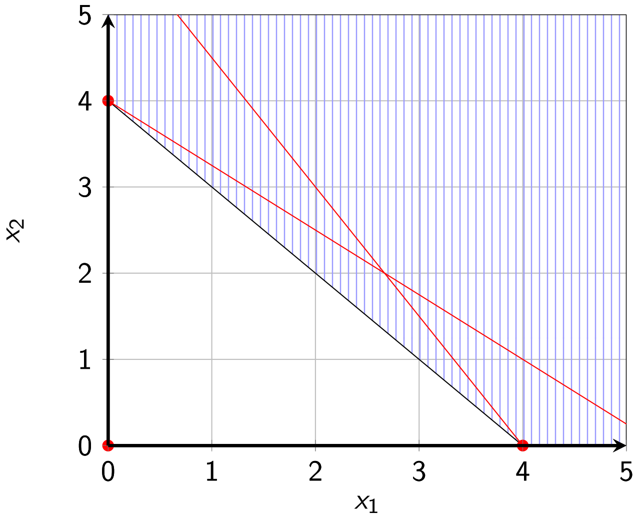
Solution
$z = 17, x_1 = 8, x_2 = 9$



Vertices are Solutions

$$x_1 + x_2 \leq 4$$
$$x_1, x_2 \geq 0$$

Maximize
$z = 1.5x_1 + x_2$
or
Maximize
$z = 0.75x_1 + x_2$

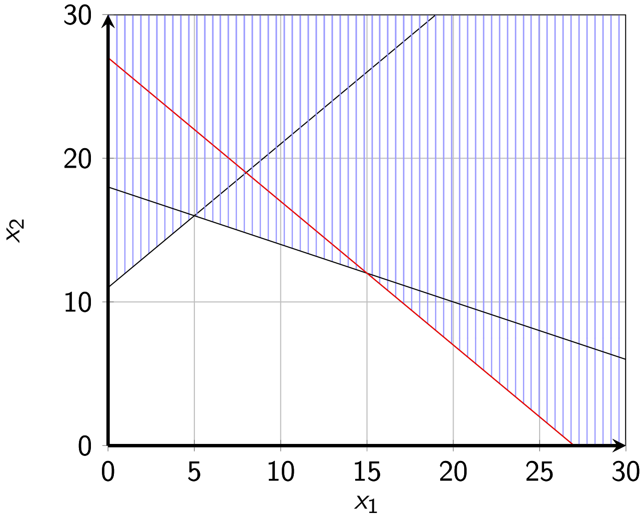


Edges End at Vertices

$$-x_1 + x_2 \leq 11$$
$$x_1 + x_2 \leq 27$$
$$2x_1 + 5x_2 \leq 90$$
$$x_1, x_2 \geq 0$$

Maximise
$z = x_1 + x_2$

Solution
$z = 27$
$x_1 = 15, x_2 = 12$



A Simplification

For the majority of this lecture we will assume $\mathbf{b} \geq 0$ e.g. our original problem did not contain $=$ or $\geq$.

We will briefly discuss how to deal with the other cases at the end.

Running Simplex on this simplified case will be Examinable. Dealing with the non-simplified case will not be, but is essential to actually implement this for real problems!

Are such Linear Programs always feasible?

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Not necesarily. A program is infeasible if some constraint forces $x_i < 0$.

What are the implications of this simplification?

In a feasible program, the origin $\mathbf{0}$ will always be a basic feasible solution

Slack Form

We will work with constraints in slack form

- All variables are restricted to be non-negative
- All constraints are equalities with constant, non-negative RHS

An inequality $x_1 + 2x_2 \leq 20$ has some slack e.g. some value that could be added to $x_1 + 2x_2$ whilst still satisfying the constraint. We can rewrite this as an equality $x_1 + 2x_2 + s_1 = 20$ where s_1 is a slack variable.

These slack variables must be non-negative. If they become 0 then the constraint has no more slack.

We will relax slack variables until they cannot be relaxed anymore

Running Example

Maximize:	$z = f(x, y) = 4x_1 + 6x_2$
Subject to:	$-x_1 + x_2 \leq 11$
	$x_1 + x_2 \leq 27$
	$2x_1 + 5x_2 \leq 90$
	$x_1 \geq 0, x_2 \geq 0$

Slack Form:

$$
\begin{aligned}
-x_1 + x_2 + s_1 &= 11 \\
x_1 + x_2 + s_2 &= 27 \\
2x_1 + 5x_2 + s_3 &= 90
\end{aligned}
$$

The Simplex Tableaux

We put all of the constraint equations into an augmented matrix (similar to Gaussian Elimination) called the Simplex Tableaux.

$$\begin{bmatrix} \mathbf{A} & \mathbf{I} & 0 \\ -\mathbf{c}^T & \mathbf{0}^T & 1 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{s} \\ z \end{bmatrix} = \begin{bmatrix} \mathbf{b} \\ 0 \end{bmatrix}$$

Note that you'll see this written in lots of different ways elsewhere

This rearranges into slack form and rewrites the objective function into the form $0 = \ldots$

For our running example it is

x_1	x_2	s_1	s_2	s_2	z	
-1	1	1	0	0	0	11
1	1	0	1	0	0	27
2	5	0	0	1	0	90
-4	-6	0	0	0	1	$\underline{0}$

(Optimal) Solutions

For each constraint row we identify a basic variable such that we imagine the constraints is written in terms of that variable e.g. it is 1 in that row and 0 elsewhere.

x_1	x_2	s_1	s_2	s_3		*Basic*
-1	1	1	0	0	11	s_1
1	1	0	1	0	27	s_2
2	5	0	0	1	90	s_3
-4	-6	0	0	0	$\underline{0}$	

$$s_1 = x_1 - x_2$$
$$s_2 = -x_1 - x_2$$
$$s_3 = -2x_1 - 5x_2$$

If we set the non-basic variables to 0 then we get a basic solution.
This is the same as starting at vertex $\mathbf{0}$

We will write this solution as $(x_1, x_2, s_1, s_2, s_3) = (0, 0, 11, 27, 90)$

The solution is not optimal as the bottom row contains coefficients < 0 e.g. we can make z bigger by increasing something

Pivoting 1

We are going to pivot to bring a non-basic variable into the solution, and necessarily remove an existing basic variable. This can be seen as squashing the slack out of a constraint.

The non-basic variable to include is the entering variable and the basic variable to remove is the departing variable.

We can pick the entering variable arbitrarily but a good heuristic is to pick the one with the smallest coefficient (why?) in the bottom row (picking the leftmost in ties avoids non-termination).

x_1	x_2	s_1	s_2	s_3		*Basic*
-1	1	1	0	0	11	s_1
1	1	0	1	0	27	s_2
2	5	0	0	1	90	s_3
-4	-6	0	0	0	$\underline{0}$	

Pivoting 2

How much can we increase this non-basic variable by? We are limited by the amount of slack. If we increase it by more than b_i then the other variables become negative (which is not allowed). Therefore, we can only increase it as much as the tightest bound allows, this gives us our departing variable.

x_1	x_2	s_1	s_2	s_3		*Basic*	Amount of Slack
-1	1	1	0	0	11	s_1	$11/1 = 11$
1	1	0	1	0	27	s_2	$27/1 = 27$
2	5	0	0	1	90	s_3	$90/5 = 18$
-4	-6	0	0	0	$\underline{0}$		

Pivoting 3

We now apply Gaussian Elimination to make our entering variable x_2 basic in the row of the departing variable.

In this case, x_2 is already 1 in the necessary row, so we just need to make it 0 elsewhere. Do you remember how to do that?

x_1	x_2	s_1	s_2	s_3		*Basic*
-1	1	1	0	0	11	s_1
1	1	0	1	0	27	s_2
2	5	0	0	1	90	s_3
-4	-6	0	0	0	$\underline{0}$	

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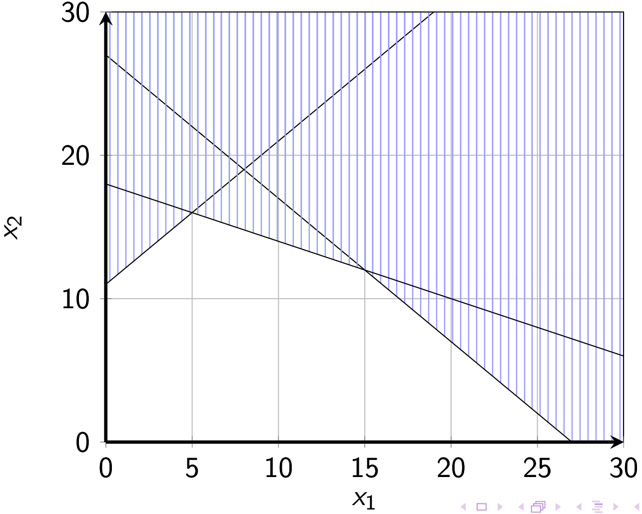
x_1	x_2	s_1	s_2	s_3		Basic	
-1	1	1	0	0	11	x_2	
2	0	-1	1	0	16	s_2	*subtract 1*
7	0	-5	0	1	35	s_3	*subtract 5*
-10	0	6	0	0	$\underline{66}$		*add 6*

New basic solution $(x_1, x_2, s_1, s_2, s_3) = (0, 11, 0, 16, 35)$ with $z = 66$.

Where are we now?

New basic solution $(x_1, x_2, s_1, s_2, s_3) = (0, 11, 0, 16, 35)$ with $z = 66$.



If we had selected x_1 instead?

x_1	x_2	s_1	s_2	s_3		Basic	Amount of Slack
-1	1	1	0	0	11	s_1	$11/-1 = -11$
1	1	0	1	0	27	s_2	$27/1 = 27$
2	5	0	0	1	90	s_3	$90/2 = 45$
-4	-6	0	0	0	$\underline{0}$		

x_1	x_2	s_1	s_2	s_3		Basic	
0	2	1	1	0	38	s_1*	*add 1*
1	1	0	1	0	27	x_1	
0	3	0	-2	1	36	s_3	*subtract 2*
0	-2	0	4	0	$\underline{108}$		*add 4*

New basic solution $(x_1, x_2, s_1, s_2, s_3) = (27, 0, 38, 0, 36)$ with $z = 66$.

Repeat

What are the entering and departing variables now?

x_1	x_2	s_1	s_2	s_3		*Basic*
-1	1	1	0	0	11	x_2
2	0	-1	1	0	16	s_2
7	0	-5	0	1	35	s_3
-10	0	6	0	0	$\underline{66}$	

Repeat

What are the entering and departing variables now? x_1 and s_3

x_1	x_2	s_1	s_2	s_3		*Basic*
-1	1	1	0	0	11	x_2
2	0	-1	1	0	16	s_2
7	0	-5	0	1	35	s_3
-10	0	6	0	0	$\underline{66}$	

Repeat

What are the entering and departing variables now? x_1 and s_3

x_1	x_2	s_1	s_2	s_3		*Basic*
-1	1	1	0	0	11	x_2
2	0	-1	1	0	16	s_2
7	0	-5	0	1	35	s_3
-10	0	6	0	0	$\underline{66}$	

So what do we do now?

Repeat

What are the entering and departing variables now? x_1 and s_3

x_1	x_2	s_1	s_2	s_3		Basic
-1	1	1	0	0	11	x_2
2	0	-1	1	0	16	s_2
7	0	-5	0	1	35	s_3
-10	0	6	0	0	<u>66</u>	

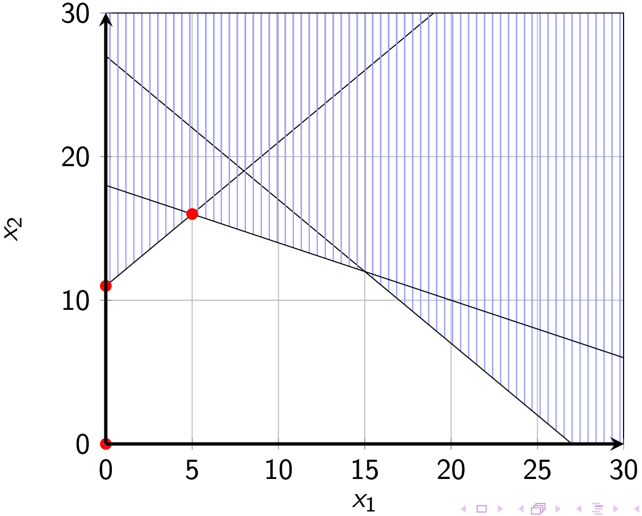
So what do we do now?

x_1	x_2	s_1	s_2	s_3		Basic
0	1	$\frac{2}{7}$	0	$\frac{1}{7}$	16	x_2
0	0	$\frac{3}{7}$	1	$-\frac{2}{7}$	6	s_2
1	0	$-\frac{5}{7}$	0	$\frac{1}{7}$	5	x_1
0	0	$-\frac{8}{7}$	0	$\frac{10}{7}$	<u>116</u>	

New basic solution $(x_1, x_2, s_1, s_2, s_3) = (5, 16, 0, 6, 0)$ with $z = 116$.

Where are we now?

New basic solution $(x_1, x_2, s_1, s_2, s_3) = (5, 16, 0, 6, 0)$ with $z = 116$.



Are we finished?

x_1	x_2	s_1	s_2	s_3		*Basic*
0	1	$\frac{2}{7}$	0	$\frac{1}{7}$	16	x_2
0	0	$\frac{3}{7}$	1	$-\frac{2}{7}$	6	s_2
1	0	$-\frac{5}{7}$	0	$\frac{1}{7}$	5	x_1
0	0	$-\frac{8}{7}$	0	$\frac{10}{7}$	$\underline{116}$	

Are we finished?

x_1	x_2	s_1	s_2	s_3		*Basic*
0	1	$\frac{2}{7}$	0	$\frac{1}{7}$	16	x_2
0	0	$\frac{3}{7}$	1	$-\frac{2}{7}$	6	s_2
1	0	$-\frac{5}{7}$	0	$\frac{1}{7}$	5	x_1
0	0	$-\frac{8}{7}$	0	$\frac{10}{7}$	$\underline{116}$	

What are the entering and departing variables now?

Are we finished?

x_1	x_2	s_1	s_2	s_3		*Basic*
0	1	$\frac{2}{7}$	0	$\frac{1}{7}$	16	x_2
0	0	$\frac{3}{7}$	1	$-\frac{2}{7}$	6	s_2
1	0	$-\frac{5}{7}$	0	$\frac{1}{7}$	5	x_1
0	0	$-\frac{8}{7}$	0	$\frac{10}{7}$	$\underline{116}$	

What are the entering and departing variables now? s_1 and s_2

Are we finished?

x_1	x_2	s_1	s_2	s_3		*Basic*
0	1	$\frac{2}{7}$	0	$\frac{1}{7}$	16	x_2
0	0	$\frac{3}{7}$	1	$-\frac{2}{7}$	6	s_2
1	0	$-\frac{5}{7}$	0	$\frac{1}{7}$	5	x_1
0	0	$-\frac{8}{7}$	0	$\frac{10}{7}$	116	

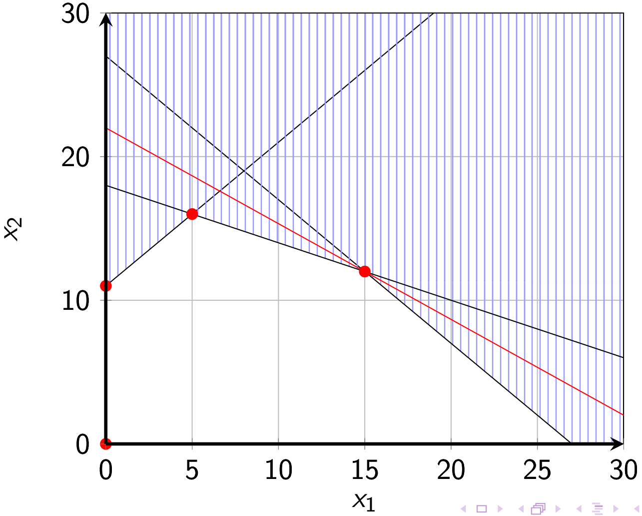
What are the entering and departing variables now? s_1 and s_2

x_1	x_2	s_1	s_2	s_3		*Basic*
0	1	0	$-\frac{2}{3}$	$\frac{1}{3}$	12	x_2
0	0	1	$\frac{7}{3}$	$-\frac{2}{7}$	14	s_1
1	0	0	$\frac{5}{3}$	$-\frac{1}{3}$	15	x_1
0	0	0	$\frac{8}{3}$	$\frac{2}{3}$	132	

New basic solution $(x_1, x_2, s_1, s_2, s_3) = (15, 12, 14, 0, 0)$ with $z = 132$.

Where are we now?

Optimal basic solution $(x_1, x_2, s_1, s_2, s_3) = (15, 12, 14, 0, 0)$ with $z = 132$.



Simplex Psuedocode

1. Introduce slack variables to get equalities
2. Form initial tableaux
3. If basic solution is optimal then **finish**
4. Identify entering and leaving variables
5. Pivot (Gaussian Elimination)
6. Go to 3

What about that Simplification?

If we did not simplify things then:

1. There may have been no solution (problem infeasible)
2. The problem may have been unbounded
3. The origin **0** is probably not a vertex

Two solutions:

1. Two-Phase Solution: first phase checks feasibility using an extended tableaux, second phase uses the result (if it is feasible) as starting point.
2. The M-Method: adds artificial variables with a massive M value to stop the algorithm exploring in certain directions.

Another Example

A company manufactures two products, A and B and wants to maximise its profit. The relevant production data is as follows:

- Profit per unit: £2 and £5 respectively
- Labour time per unit: 2 hours and 1 hour respectively
- Machine time per unit: 1 hour and 2 hours respectively
- Available labour and machine time: 80 hours and 65 hours respectively

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- Machine time per unit: 1 hour and 2 hours respectively
- Available labour and machine time: 80 hours and 65 hours respectively
- Labour and machine overtime cost: £10 and £2 per hour, respectively

Passing on a Message

I need to get a message from my office to all other offices in the building. The message can be passed on by somebody in one office going to another office and telling them the message. Each person only knows where some offices are and nobody wants to visit more than one office; they will go to another office and then back to their own. These trips will will waste some time (e.g. they won't be working) but some people gossip more than others so will waste different amounts of time.

I know

- Who is in which office $(pA, o1), (pB, o2), \ldots$
- Which offices a person knows about $(pA, o2), (pA, o3), (pB, o1) \ldots$
- How much time each person will waste $(pA, 4), (pB, 2), \ldots$

What's the least amount of time I need to waste?