Algorithmic Techniques

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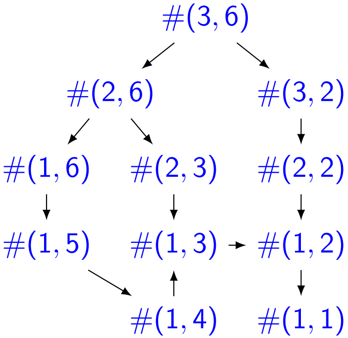
  - edge cases: $i = 1$, $a = w_i$, $a < w_i$, $a = 0$

Overlapping Subproblems

$$\#(i, a) = \min(\#(i-1, a), 1 + \#(i, a - w_i)) \qquad \text{if } i > 1 \land a > w_i$$

- Overlapping subproblems: Eg make 6 in coin system 1-3-4



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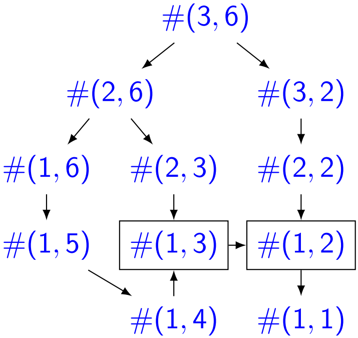

Algorithm (Variant 1)

compute all subproblems in suitable order

```
procedure NUM($a$)
    if $a = 0$ then return $0$
    $r \leftarrow$ new array $n \times a$
    for $i = 1 \dots n$ do
        for $k = 1 \dots a$ do
            if $k = w_i$ then
                $r[i, k] \leftarrow 1$
            else if $k < w_i$ then
                $r[i, k] \leftarrow r[i - 1, k]$
            else if $i = 1$ then
                $r[i, k] \leftarrow 1 + r[i, k - w_1]$
            else
                $r[i, k] \leftarrow \min(r[i - 1, k], 1 + r[i, k - w_1])$
    return $r[n, a]$
```

Algorithm (Variant 2)

memorize already computed subproblems

```
global map $r \leftarrow$ empty map
procedure NUM_AUX($i, k$)
    if not defined $r[i, k]$ then
        if $k = w_i$ then
            $r[i, k] \leftarrow 1$
        else if $k < w_i$ then
            $r[i, k] \leftarrow \text{NUM\_AUX}(i - 1, k)$
        else if $i = 1$ then
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        else
            $r[i, k] \leftarrow \min(\text{NUM\_AUX}(i - 1, k), 1 + \text{NUM\_AUX}(i, k - w_1))$
    return $r[i, k]$
procedure NUM($a$)
    if $a = 0$ then return $0$
    return NUM_AUX(n,a)
```

Properties Required for Dynamic Programming

- Optimal substructure
- Overlapping subproblems
    - otherwise, simple recursion would be sufficient!

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$$d(u, v, \{w\} \cup X) = \min(d(u, w, X) + d(w, v, X), d(u, v, X))$$

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```
procedure FLOYD__WARSHALL
    $dist[u, v] \leftarrow w(u, v)$ for all nodes $u$,$v$
    $dist[v, v] \leftarrow 0$ for all nodes $v$
    for $w \in V$ do            // Compute $d(u, v, X)$ for increasing set $X$
        for $u \in V$ do
            for $v \in V$ do
                $dist[u, v] \leftarrow \min(dist[u, v], dist[u, w] + dist[w, v])$
```

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- Application: diff - tool

w_1 `dist[i,j] = dist[k,i] + dist[j,k] + 1`

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$lcs(w_1, w_2)$ `dist[i,j] = dist[,] + dist[,]`

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  - letters in w_1 but not LCS: removed
  - letters in w_2 but not LCS: added

LCS with Dynamic Programming

Optimal substructure

$$\begin{aligned} lcs(w_1x, w_2y) =& lcs(w_1, w_2)x && \text{if } x = y \\ =& \max(lcs(w_1x, w_2), lcs(w_1, w_2y)) && \text{otherwise} \end{aligned}$$

- max longer of the two sequences. Any if equal length.
- Idea: cases if last letter of each word is part of LCS

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    - good for general CF-grammars.
    - much better algorithms for special grammars, like computer languages!

Conclusion

- Optimal Substructure
  - solve larger instance by smaller instances
  - often requires generalization, e.g.
    - only use coins $1 \dots i$
    - only use paths over certain intermediate nodes
- Greedy Choice: distinct subproblems, no need to backtrack
- Overlapping Subproblems: memorize already computed solutions