NP-Completeness

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- Example: general sorting. $O(n \log n)$
  - no faster algorithm exists!
- But how about factorization?
  - find prime factors of n bit integer
  - naive algorithm: try all possible factors. $O(2^n)$
  - fastest algorithm: unknown!

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- Recall: we count steps, wrt. some computational model
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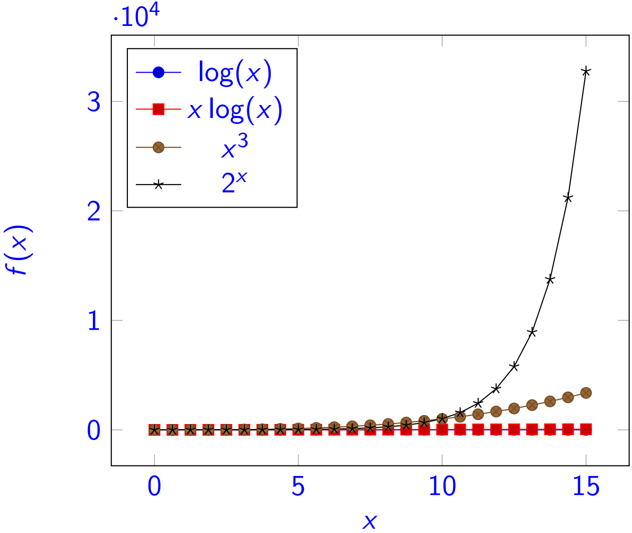
  - Naive algorithm: try all 2^n possibilities. $O(2^n)$

Comparing Execution Times



Comparing Execution Times

- If one step is 1ns (1GHz clock), and algorithm needs $f(n)$ steps
- What input size can we handle in ... (approx values)

$f(n)$	1 second	1 hour	1 day	1 year	1 century	lifetime of universe
$\log(n)$	$10^{301029995}$	$\dots$				
n	10^9	10^{12}	10^{14}	10^{16}	10^{18}	10^{26}
n^2	10^4	10^6	10^7	10^8	10^9	10^{13}
n^5	60	300	600	2000	5000	10^5
2^n	29	41	46	54	61	88

A few numbers for comparison (approx/estimates):

- 10^{78} – 10^{82}: atoms in the known universe
- $8.8 * 10^{27}$ meters: diameter of known universe
- $5 * 10^{22}$ stars in known universe
- $4 * 10^{17}$ seconds since big bang
- 10^{13} bits of memory on a hard disk (1TiB)
- 10^{11} people ever lived
- $7.8 * 10^9$ people on earth
- $6.6 * 10^6$ people in UK

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- Notation $poly(n)$: some polynomial in n

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- There are problems that can only be solved in exponential time
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- There are many problems for which we don't know
  - factorization, SAT, ...

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- We focus on decision problems here!

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  - ... graphs without negative-weight cycles
  - ... CNF-formulas that have a solution

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- Certificate checker $\mathit{check} : (\{0,1\}^*, \{0,1\}^*) \to \{0,1\}$
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- Example: does CNF have solution?
  - certificate is solution (list of variables that are true)
  - certificate checker: evaluate formula

The Complexity Class NP

Definition (NP)

A decision problem is in NP, iff it has polynomial-time checkable certificates

- for word w of size n
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  - $check(w, c)$ executable in time $poly(n)$

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- Example: CNF
  - valuation of variables: polynomial size in formula
  - evaluating formula: polynomial time in formula size
- all polynomial-time solvable problems are in NP
  - the certificate is always empty, and *check* solves the problem

Reduction

Intuitively:

- Convert problem A into problem B, in polynomial time
  - if B is in $O(f(n))$, then A is in $O(poly(n) + f(poly(n)))$
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Formally:

Definition (polynomial-time reducible)

A language L_A is *polynomial-time reducible* to L_B, iff there is a function $f : \{0, 1\}^* \rightarrow \{0, 1\}^*$ such that

1. $w \in L_A \iff f(w) \in L_B$
2. f can be computed in polynomial time

- Notation: $L_A \leq_p L_B$

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Theorem

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Proof.

- $check_A(w, c) = check_B(f(w), c)$ is certificate checker.
- and executable in polynomial time.
- note: $f(w)$ can increase word size only polynomially.

□

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- Intuition: harder (or equally hard) than any problem in NP
- NP-complete = In NP and NP-hard
- Many interesting problems turn out to be NP-complete

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- If one NP-complete problem A has poly-time algorithm then all problems B in NP have poly-time algorithm!
    - reduce B to A (in poly-time), then solve A

Complexity of NP-complete Problems

- Naive algorithm: for word of size n, enumerate all $2^{poly(n)}$ certificates, and check each in polynomial time.
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    - can you live with approximation? (for optimization problems)

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- Now: obtaining an initial NP-hard problem.