Graph Algorithms

Peter Lammich

11. Februar 2020

Outline

Outline

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1. Directed Graphs
2. Graph Traversal Algorithms
3. Shortest Path in Weighted Graphs
   - Bellman Ford Algorithm
   - Dijkstra's Algorithm
4. A* Algorithm

Dijkstra's Algorithm

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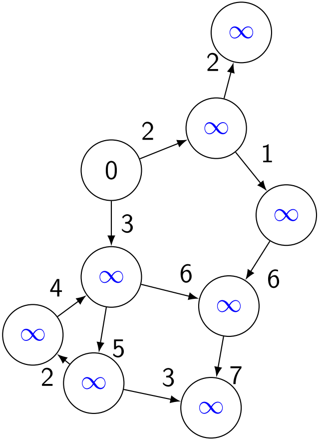
procedure $\textsc{relax}(u)$
    for all v with $w(u, v) \neq \infty$ **do** $\textsc{relax}(u, v)$
procedure $\textsc{dijkstra}(s)$
    $F \leftarrow \emptyset$, $D \leftarrow \textsc{initEstimate}(s)$
    while $V \setminus F \neq \emptyset$ **do**
        $u \leftarrow \text{Some } u \in V \setminus F, D(u) \text{ minimal}$
        $F \leftarrow F \cup \{u\}$
        $\textsc{relax}(u)$
    return D

Example



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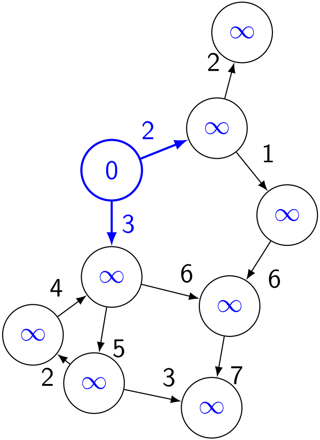

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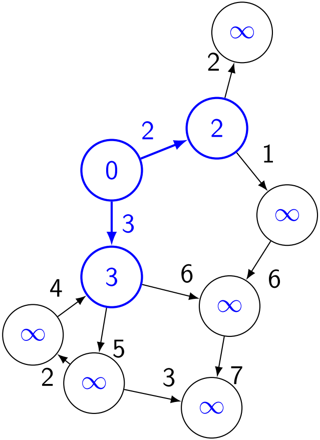

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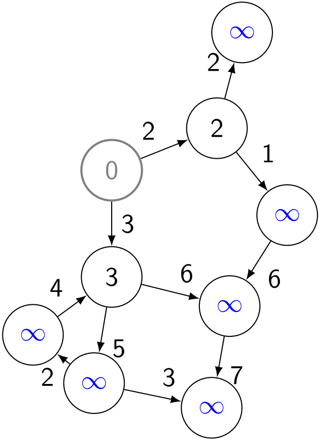

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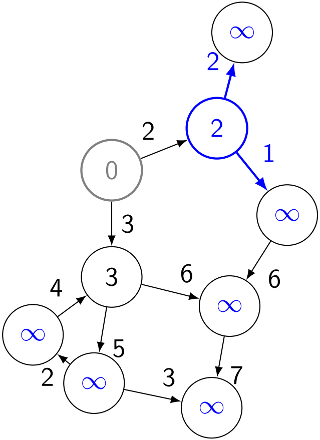

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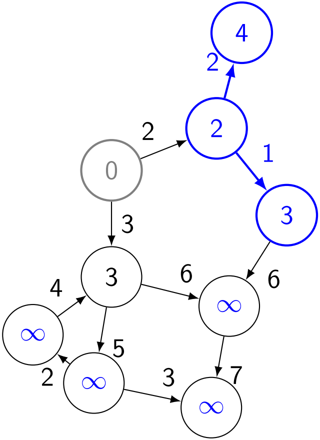

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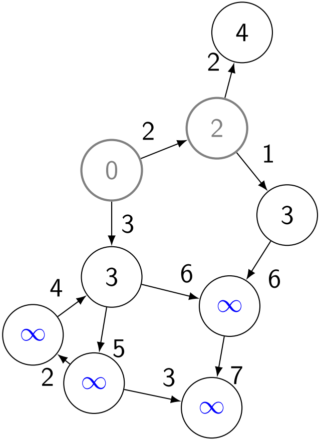

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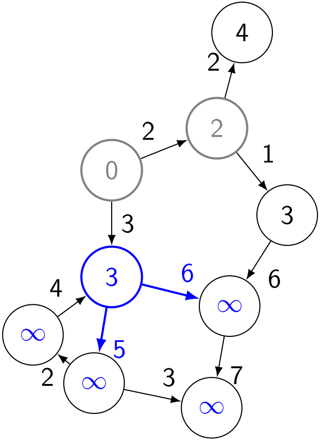

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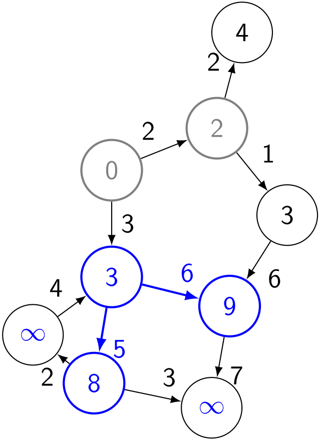

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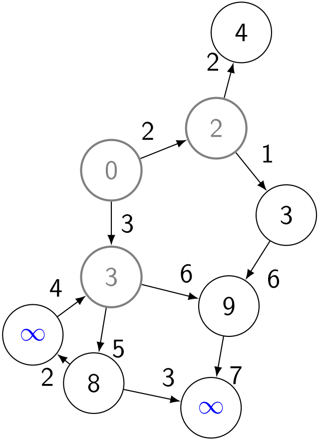

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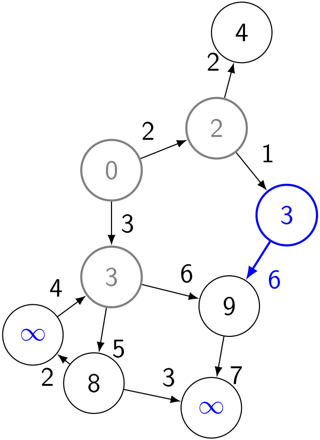

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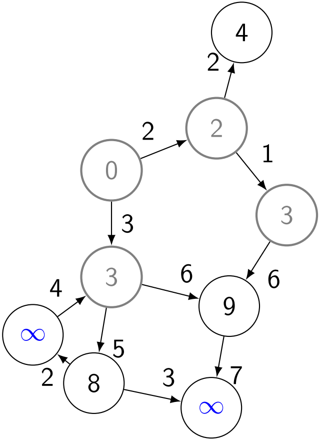

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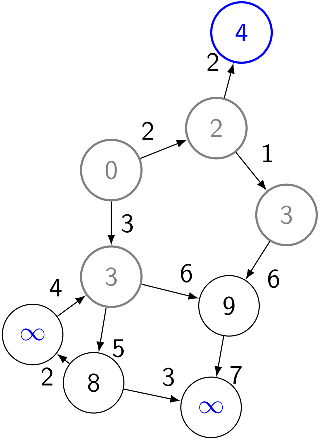

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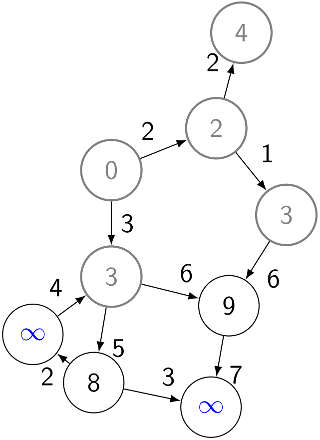

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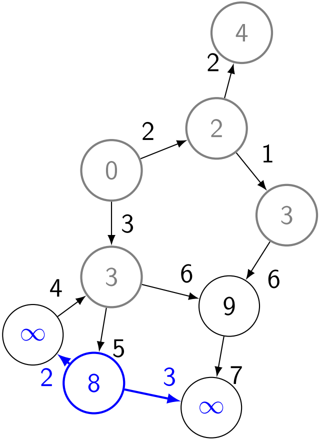

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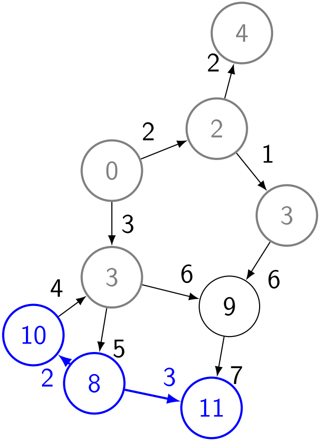

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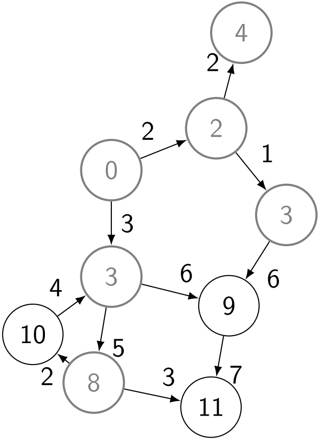

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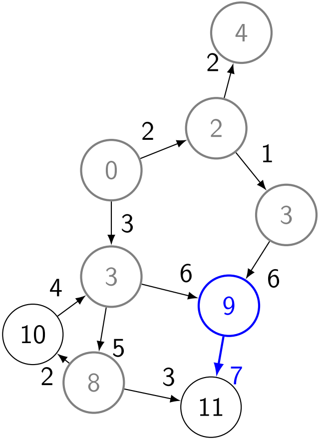

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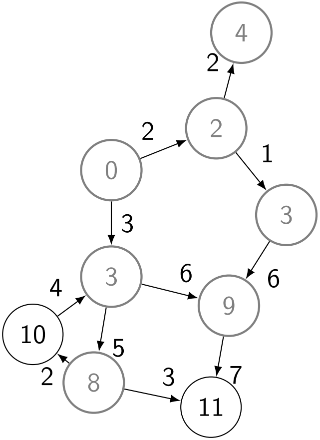

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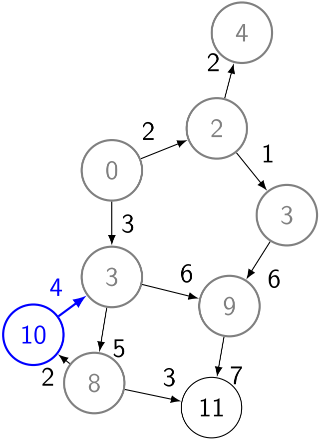

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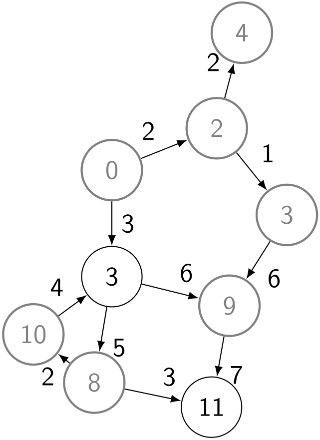

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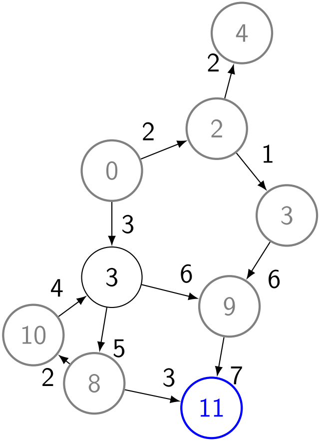

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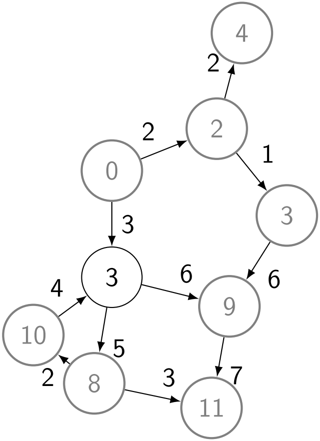

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- As invariant: For all $u \in F$
  - $D(u) = \delta(u)$ (precise)
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- Further iterations: See next slide!
- Finally: $F \supseteq V$, thus D precise for all nodes!

Dijkstra's Algorithm: Invariant preservation

Assume $s \in F$ and for all $u \in F$

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- Use predecessor map to compute actual paths

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- To find index of node in heap:
  - maintain map from node names to index in heap!

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  - recall: Dijkstra relaxes node with minimal $D(u)$

Pseudocode

procedure $\textsc{astar}(s, t)$
    $F \leftarrow \emptyset$, $D \leftarrow \textsc{initEstimate}(s)$
    while $V \setminus F \neq \emptyset$ **do**
        $u \leftarrow$ Some $u \in V \setminus F$, $D(u) + h(u)$ minimal
        $P \leftarrow F \cup \{u\}$
        if $u = t$ **then return** $D(t)$
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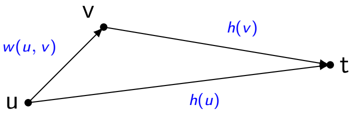

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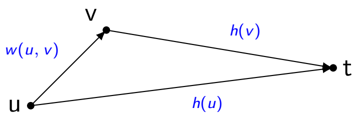

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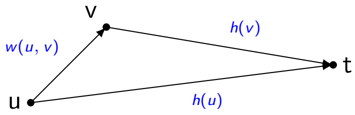

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