Graph Algorithms

Peter Lammich

3. Februar 2020

Outline

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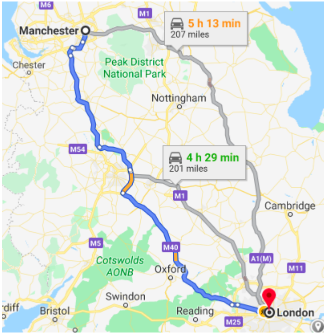

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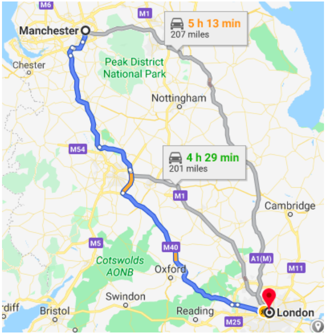
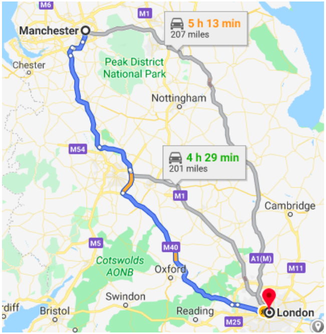
- Shortest route between Manchester and London?
- Model map as graph: Nodes=Cities, Edges=Roads
    - Label each road with length (estimated travel time, ...)



Motivation

- Shortest route between Manchester and London?
- Model map as graph: Nodes=Cities, Edges=Roads
  - Label each road with length (estimated travel time, ...)
- Compute *shortest path* between two nodes



Formal Stuff

- Graph as *weight matrix* w
  - $E = \{(u, v) \mid w(u, v) \neq \infty\}$
  - $w(u, v) = d$ — Edge between u and v has weight d
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- Recall: path p is list of nodes.
  - Path between u and v: upv
  - *Weight* of path: $|u_1 \ldots u_n| := \sum_{i=1}^{<n} w(u_i, u_{i+1})$
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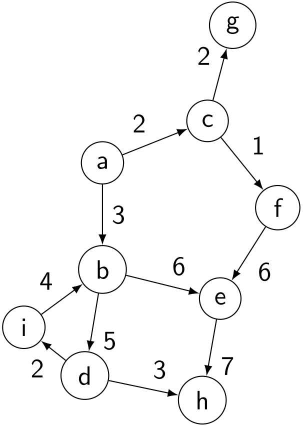
  - $|p| = \infty$ means path p not feasible!
- $\delta(u, v)$ — distance between u and v
  - $\delta(u, v) := \min |upv| \text{for all paths } p$
  - $\delta(u, v) = \infty$ — No path from u to v!

Example



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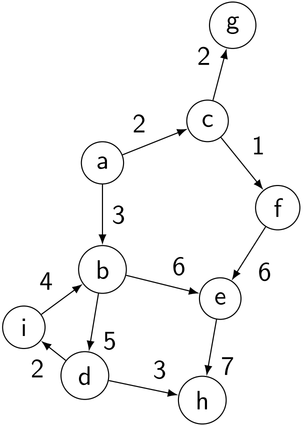

Weight Matrix

	a	*b*	*c*	*d*	*e*	*f*	*g*	*h*	*i*
a	∞	3	2	∞	∞	∞	∞	∞	∞
b	∞	∞	∞	5	6	∞	∞	∞	∞
c	∞	∞	∞	∞	∞	1	2	∞	∞
d	∞	∞	∞	∞	∞	∞	∞	3	2
e	∞	∞	∞	∞	∞	∞	∞	7	∞
f	∞	∞	∞	∞	6	∞	∞	∞	∞
g	∞	∞	∞	∞	∞	∞	∞	∞	∞
h	∞	∞	∞	∞	∞	∞	∞	∞	∞
i	∞	4	∞	∞	∞	∞	∞	∞	∞

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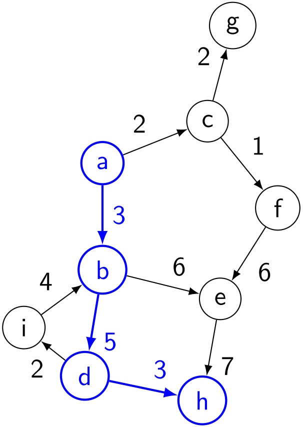

Shortest path: *abdh*
Weight: $|abdh| = 11$
Distance: $\delta(a, h) = 11$ (weight of shortest path)

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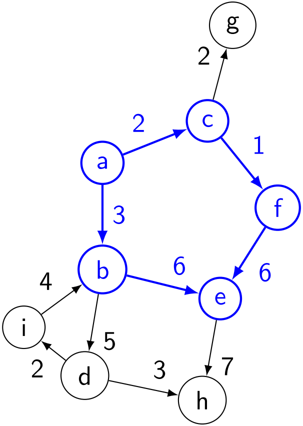

Shortest path not always unique:
Shortest paths: *acfe*, *abe*
But distance is!
Weight: $|acfe| = |abe| = 9$
Distance: $\delta(a, h) = 9$ (weight of shortest path)

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- Compute shortest paths from *start node s* to any node
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  - For edge (u, v) (*relax edge*)

procedure RELAX(u, v)
  if $D(u) + w(u, v) < D(v)$ **then**
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- Now: Strategies of relaxing edges, to reach precise estimate

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Assume no negative weight cycles exist. Negative weights are allowed!

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    $D(u) \leftarrow \infty$ for all u
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procedure $\textsc{bellmanFord}(s)$
    $D \leftarrow \textsc{initEstimate}(s)$
    for $i \in 0.. < |V| - 1$ **do**
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- Relax each edge. Repeat (at most) $|V| - 1$ times.
    - If a round changes nothing, we can stop!

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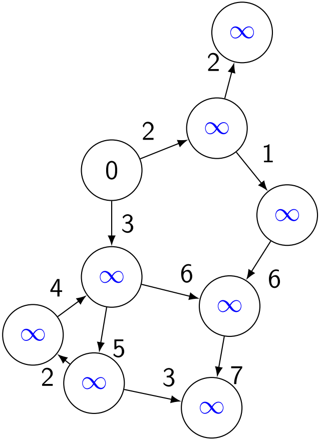
  - All shortest paths have length at most $|V| - 1$
  - Thus, D precise after algorithm

Example



In each round, relax every edge once.

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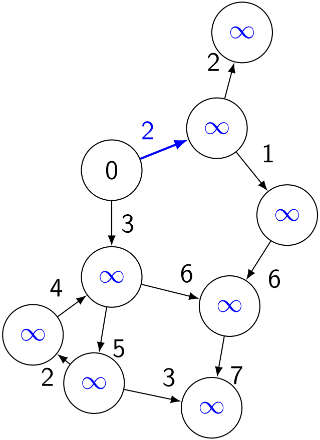

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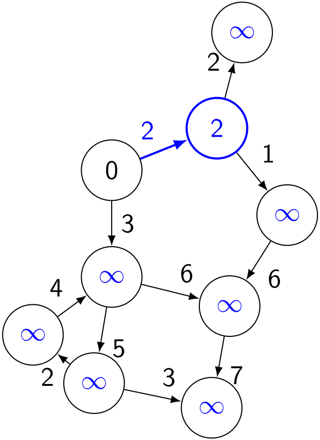

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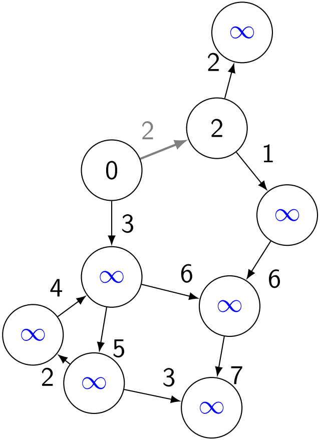

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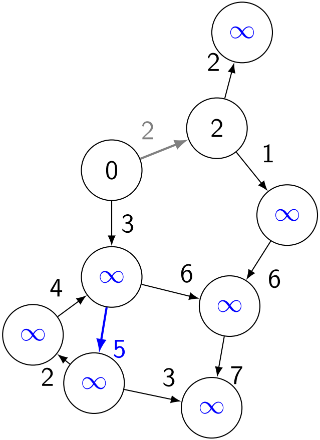

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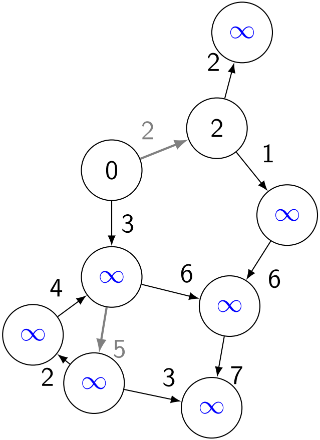

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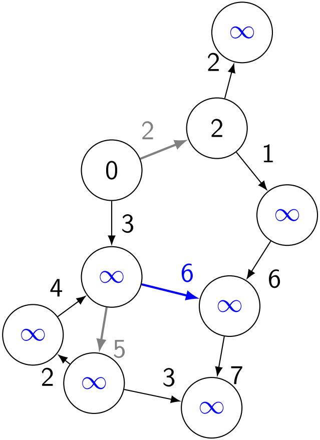

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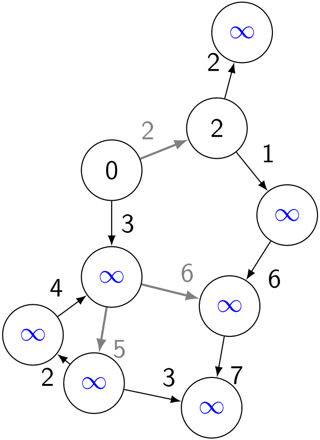

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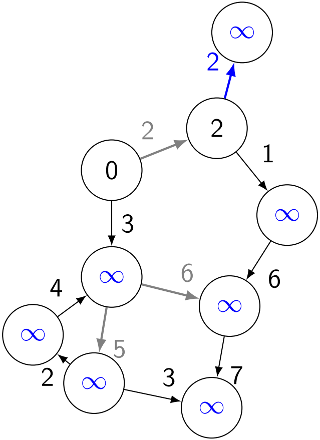

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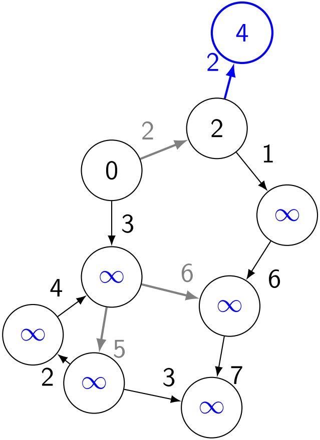

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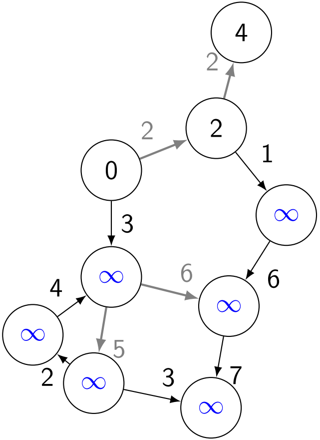

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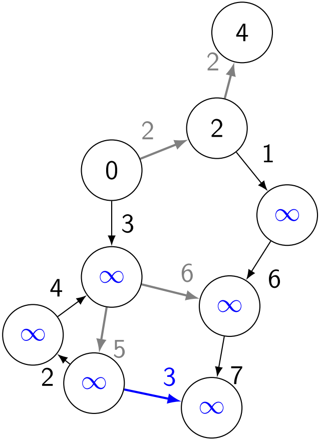

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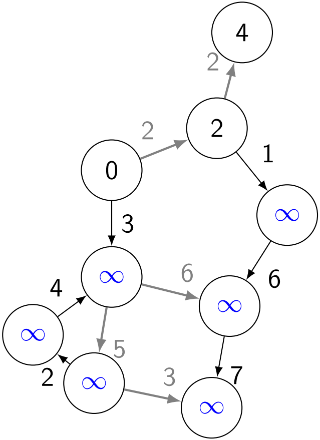

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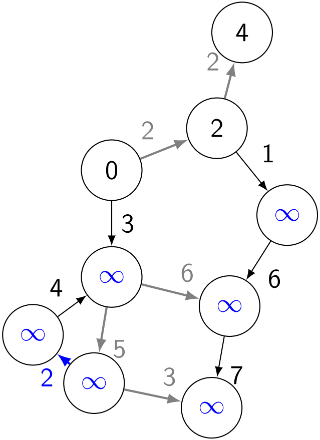

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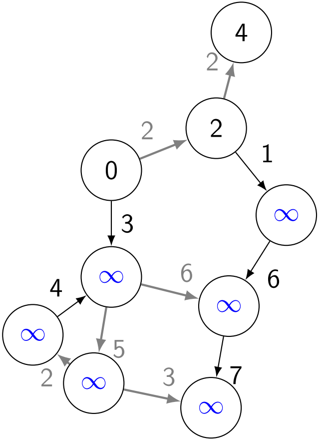

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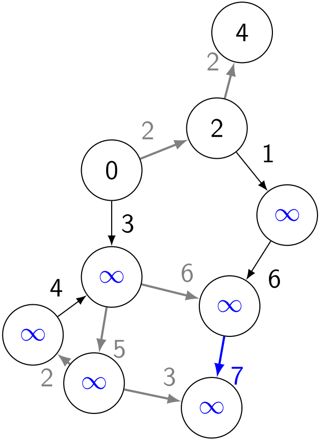

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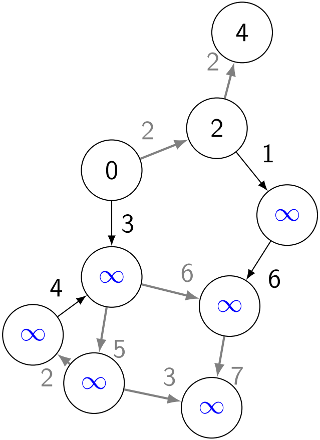

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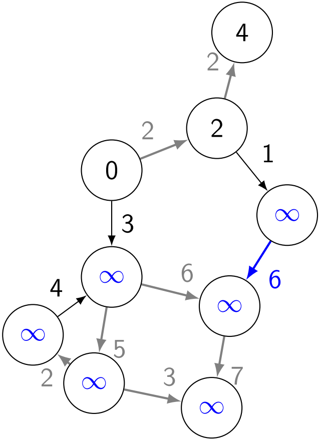

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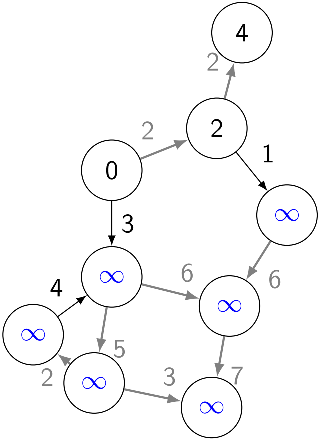

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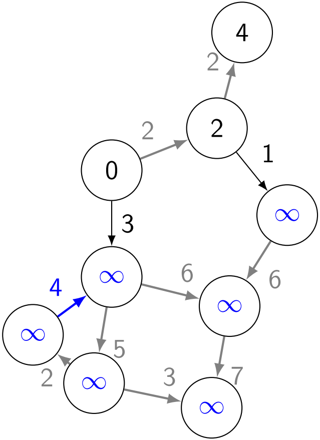

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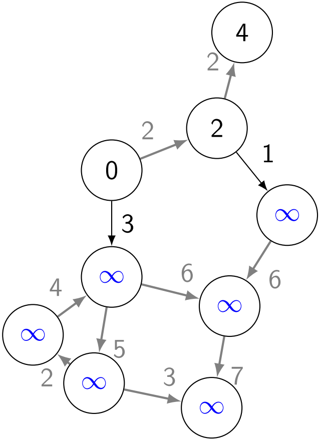

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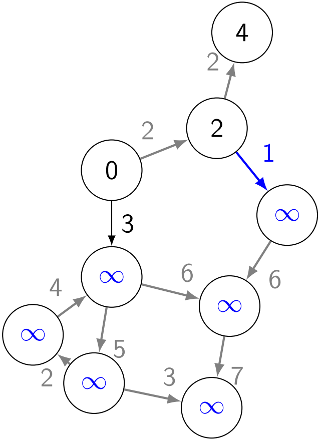

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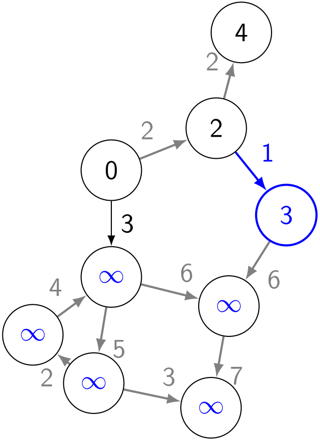

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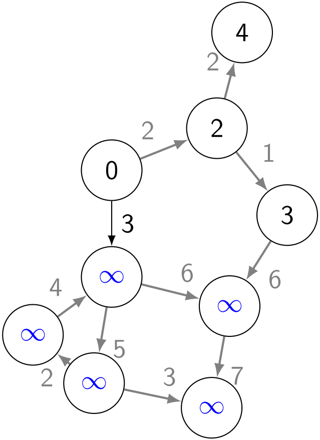

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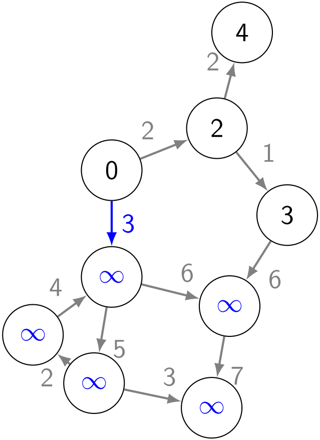

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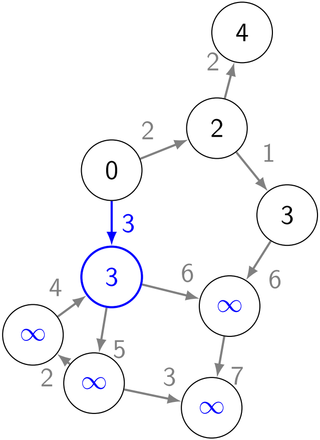

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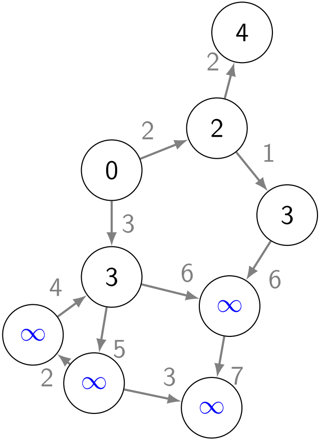

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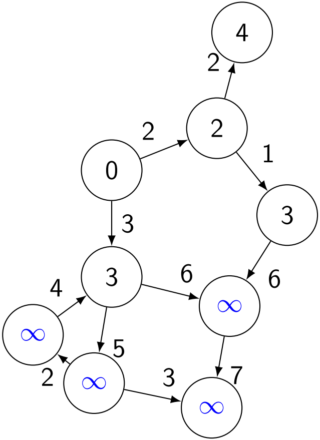

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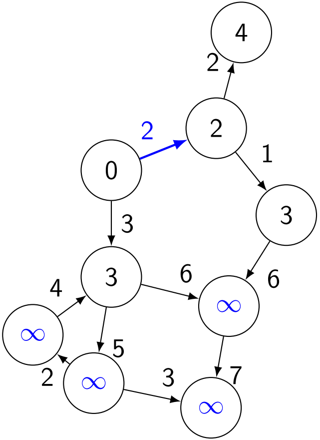
Next round ...

Example



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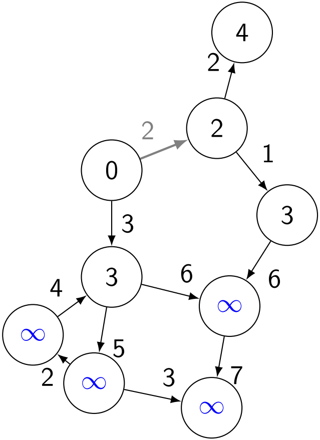

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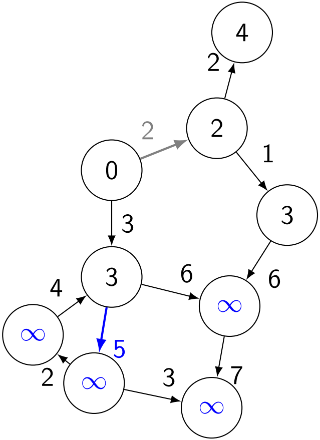

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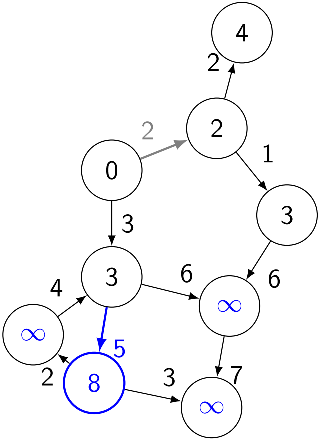

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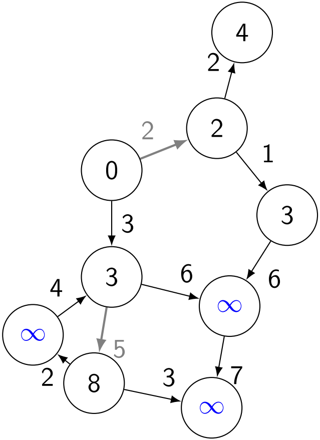

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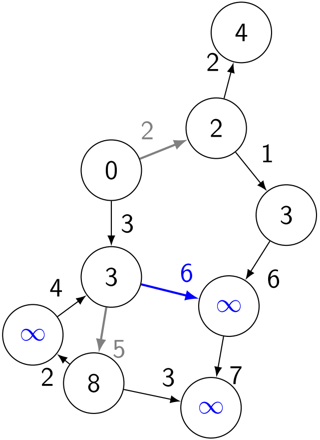

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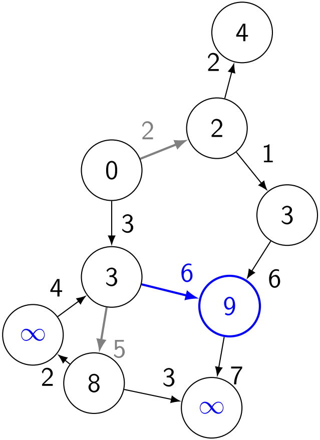

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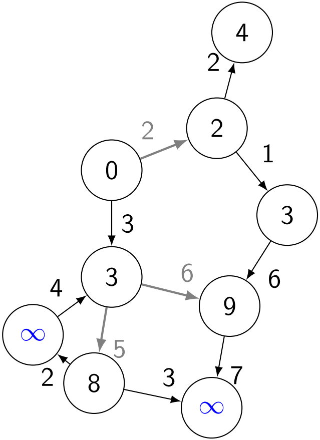

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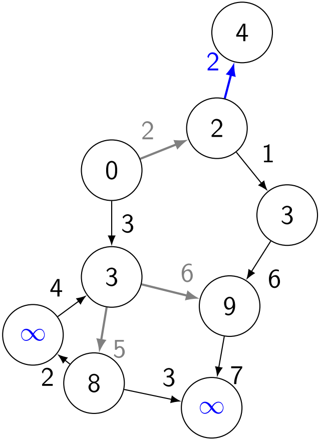

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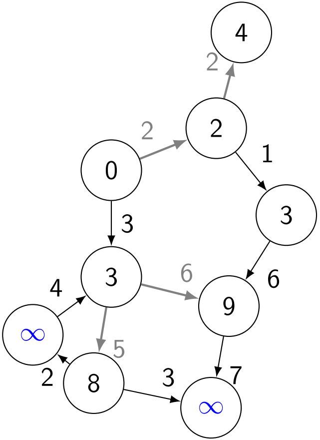
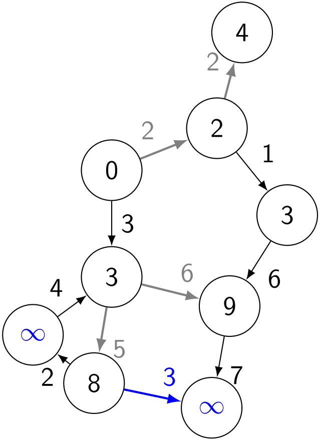
Example



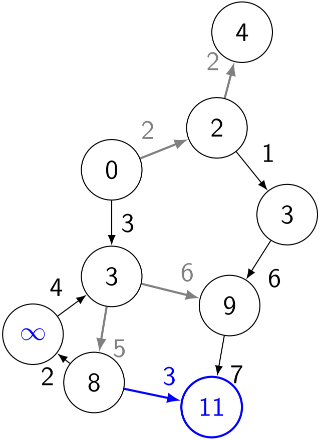
In each round, relax every edge once.

Example



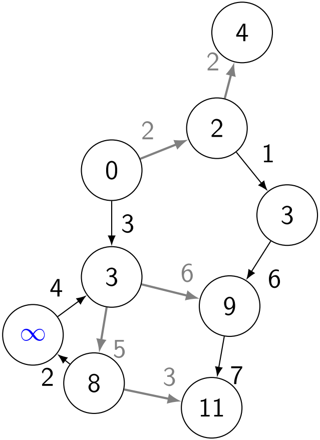
In each round, relax every edge once.

Example



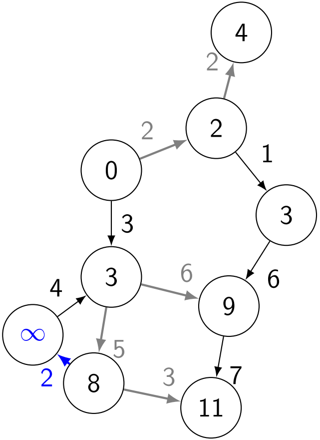
In each round, relax every edge once.

Example



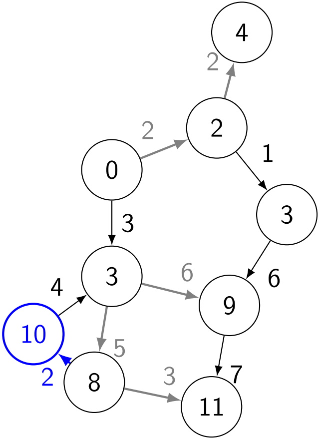
In each round, relax every edge once.

Example



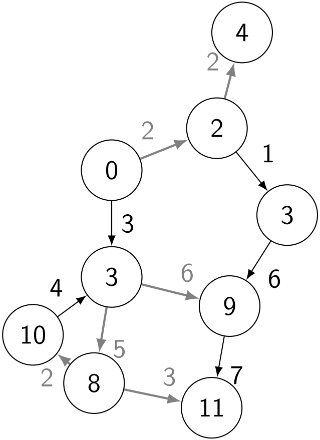
In each round, relax every edge once.

Example



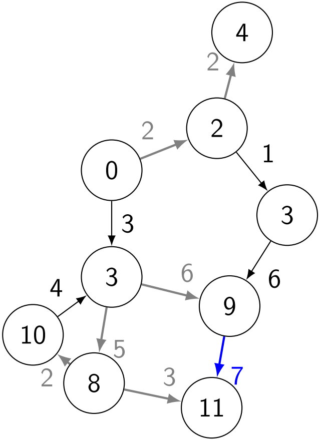
In each round, relax every edge once.

Example



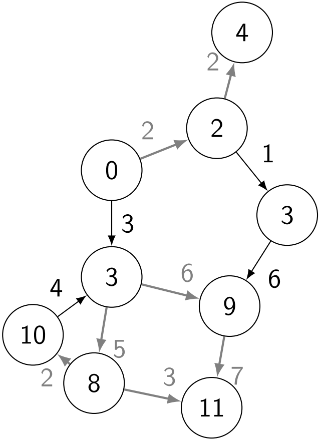
In each round, relax every edge once.

Example



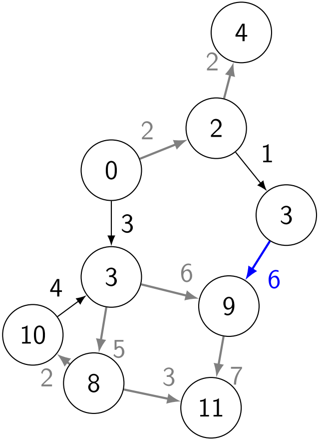
In each round, relax every edge once.

Example



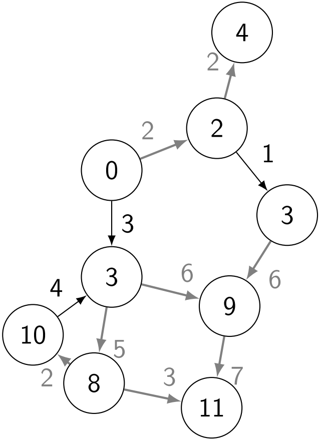
In each round, relax every edge once.

Example



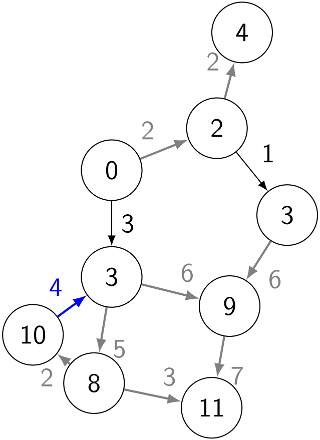
In each round, relax every edge once.

Example



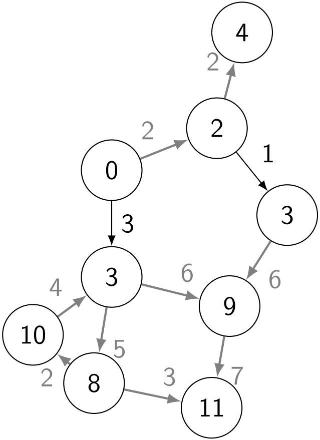
In each round, relax every edge once.

Example



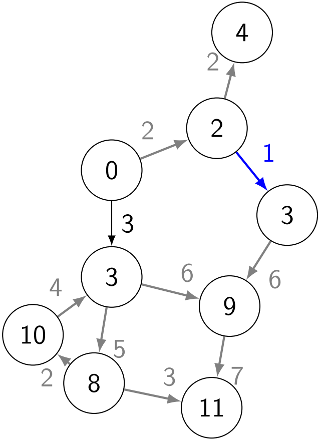
In each round, relax every edge once.

Example



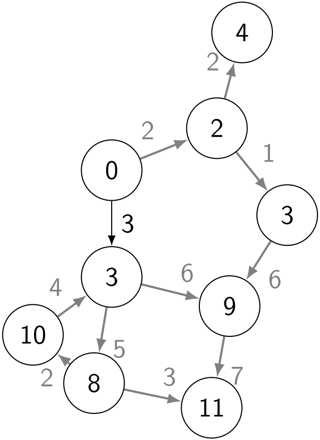
In each round, relax every edge once.

Example



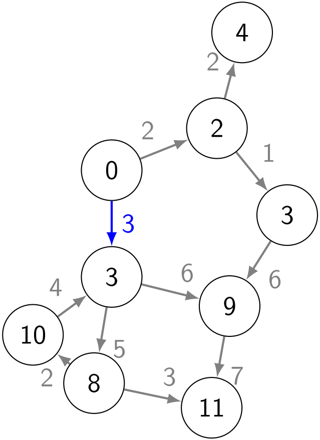
In each round, relax every edge once.

Example



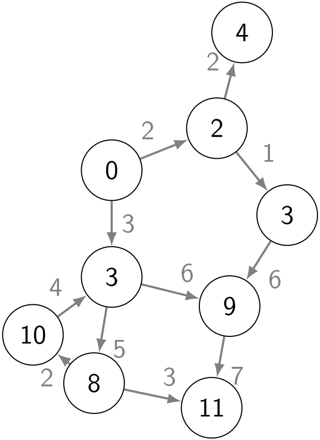
In each round, relax every edge once.

Example



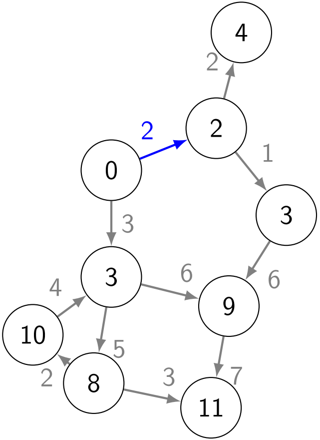
In each round, relax every edge once.

Example



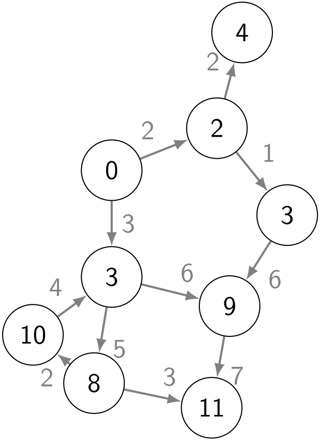
In each round, relax every edge once.
Next round ...

Example



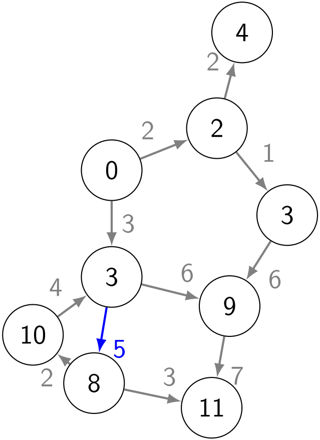
In each round, relax every edge once.

Example



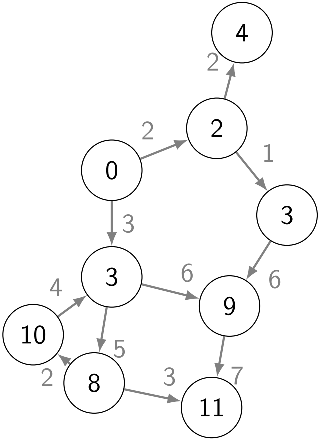
In each round, relax every edge once.

Example



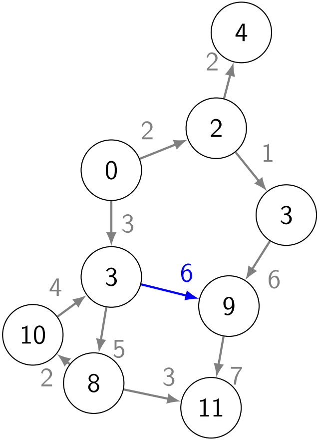
In each round, relax every edge once.

Example



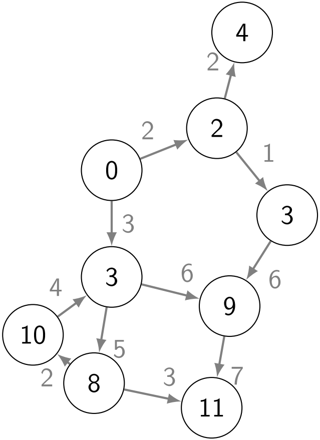
In each round, relax every edge once.

Example



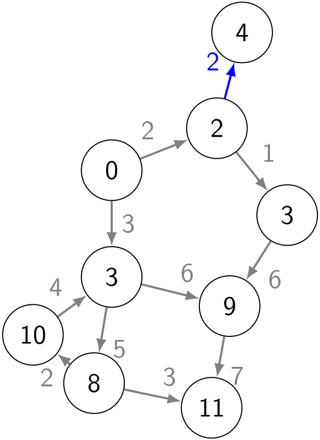
In each round, relax every edge once.

Example



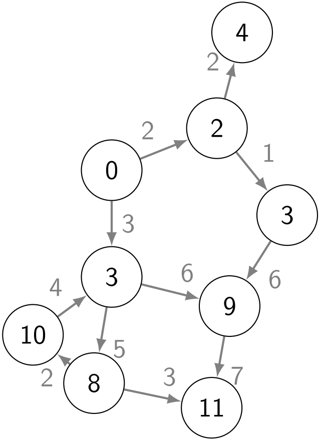
In each round, relax every edge once.

Example



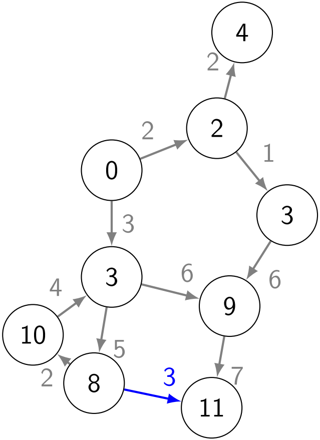
In each round, relax every edge once.

Example



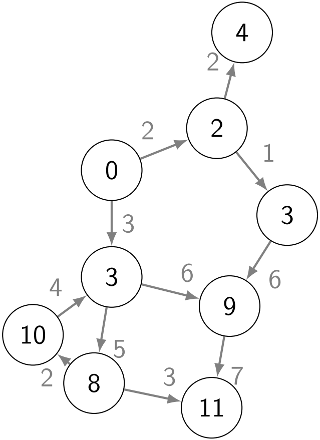
In each round, relax every edge once.

Example



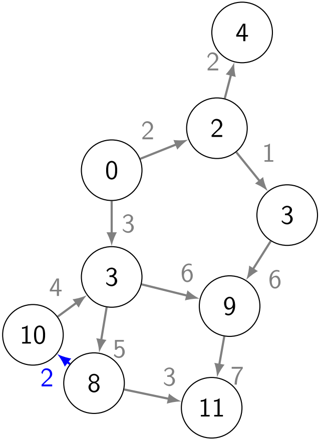
In each round, relax every edge once.

Example



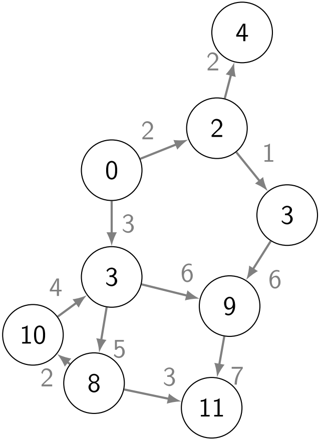
In each round, relax every edge once.

Example



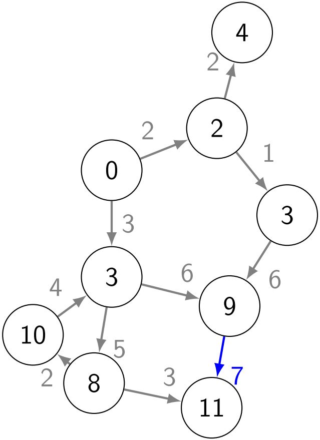
In each round, relax every edge once.

Example



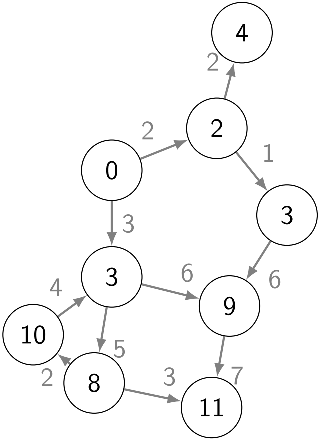
In each round, relax every edge once.

Example



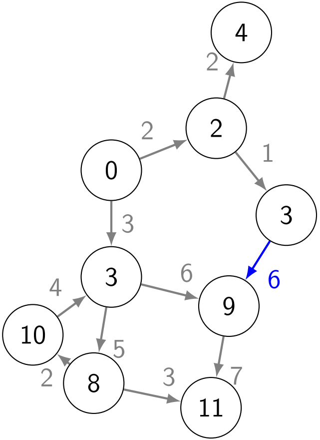
In each round, relax every edge once.

Example



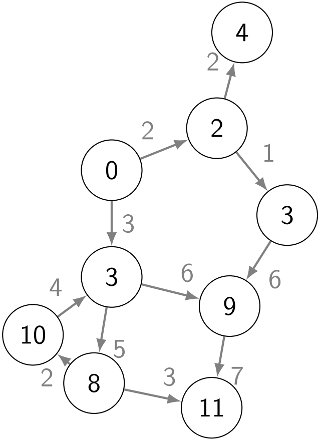
In each round, relax every edge once.

Example



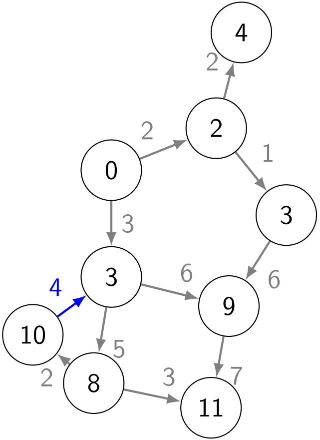
In each round, relax every edge once.

Example



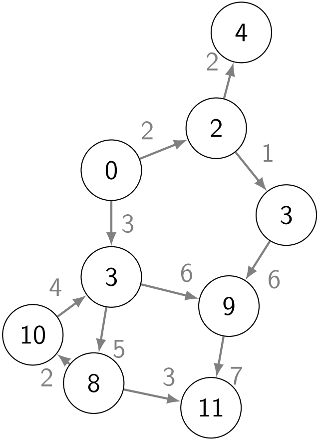
In each round, relax every edge once.

Example



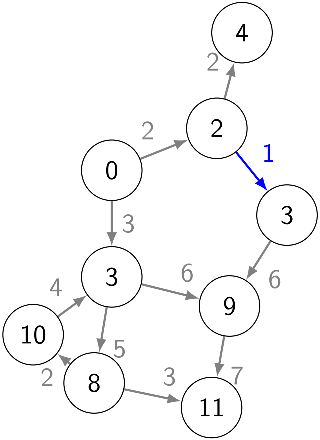
In each round, relax every edge once.

Example



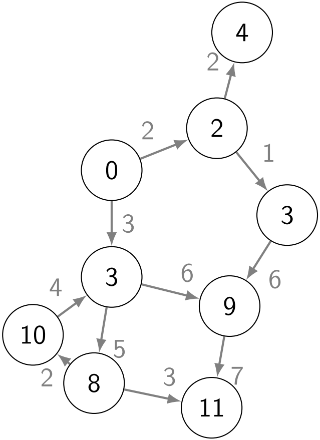
In each round, relax every edge once.

Example



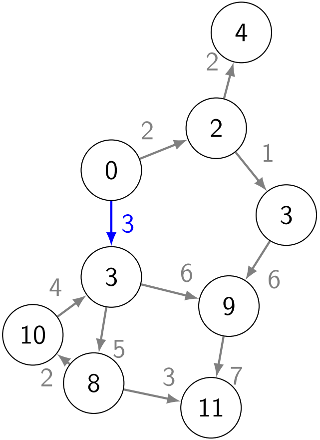
In each round, relax every edge once.

Example



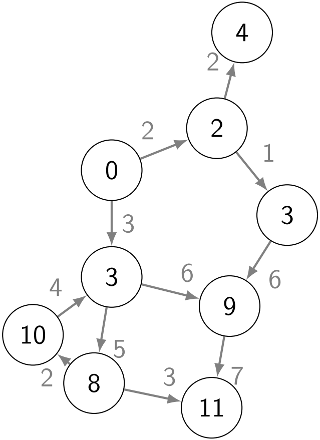
In each round, relax every edge once.

Example



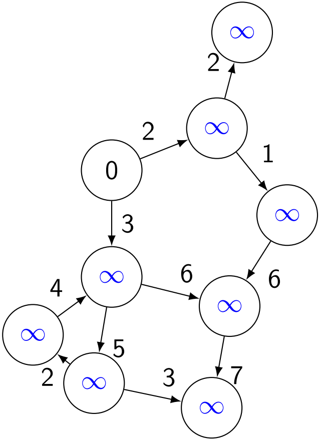
In each round, relax every edge once.

Example



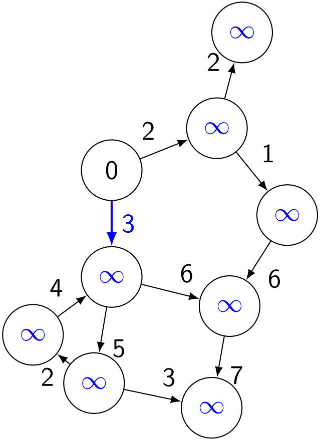
In each round, relax every edge once.
Round changed nothing: terminate

Example



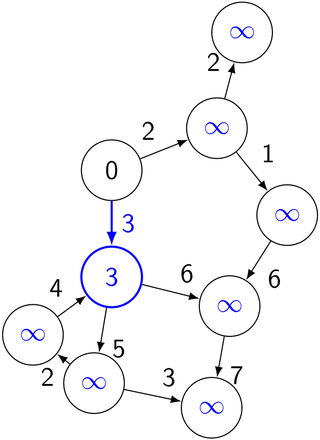
Order of edges can affects number of required rounds!

Example



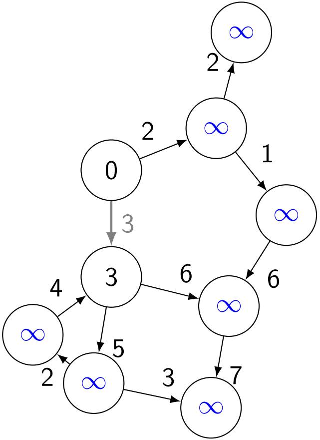
Order of edges can affects number of required rounds!

Example



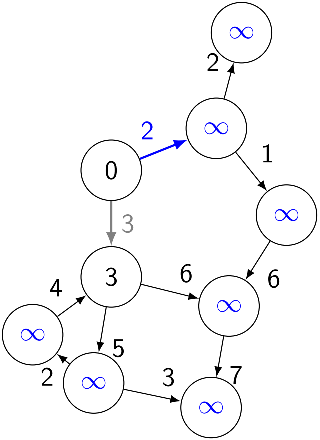
Order of edges can affects number of required rounds!

Example



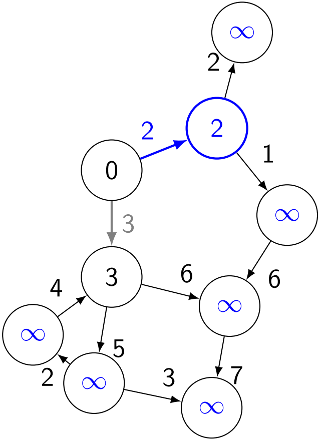
Order of edges can affects number of required rounds!

Example



Order of edges can affects number of required rounds!

Example



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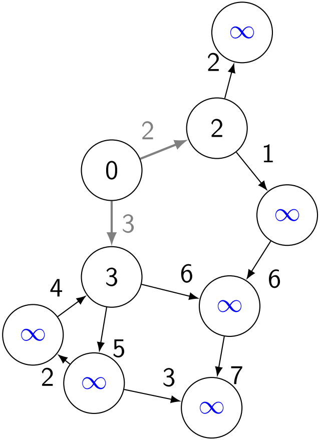
Example



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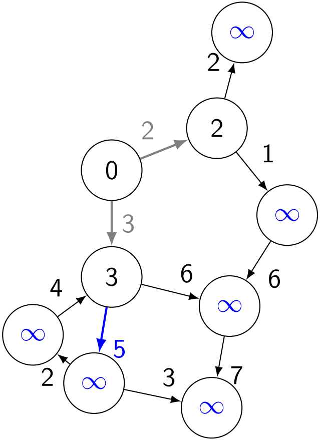
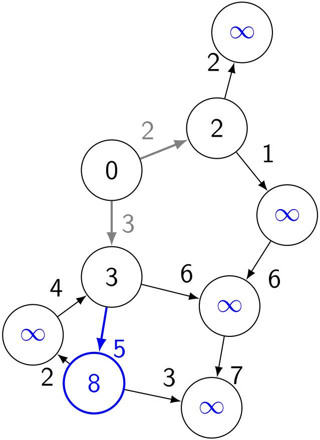
Example



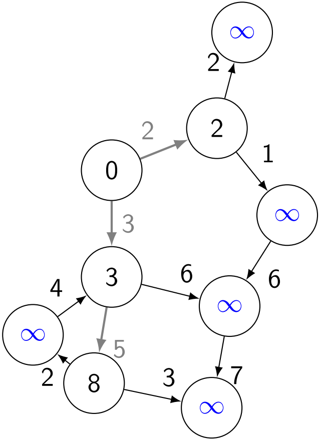
Order of edges can affects number of required rounds!

Example



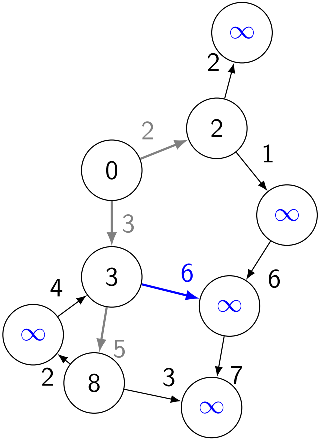
Order of edges can affects number of required rounds!

Example



Order of edges can affects number of required rounds!

Example



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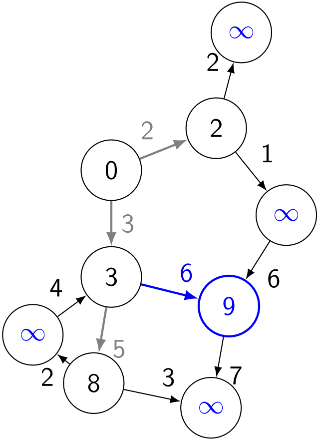
Example



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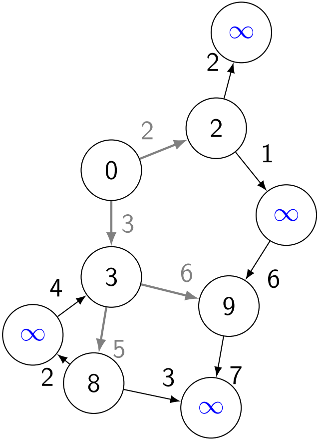
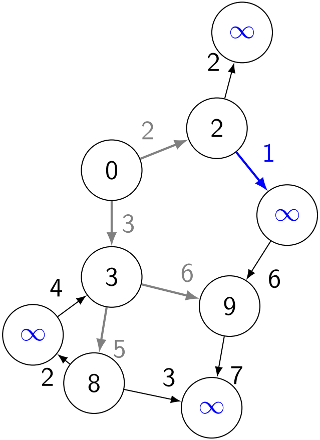


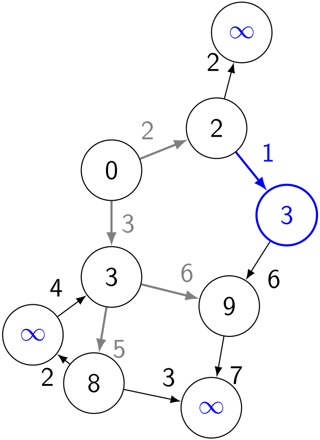
Order of edges can affects number of required rounds!

Example



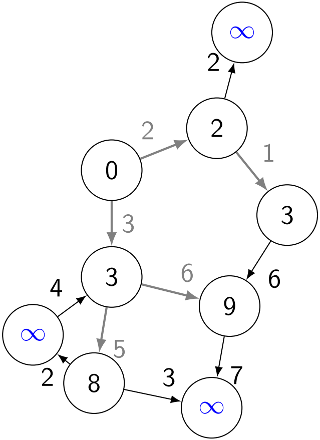
Order of edges can affects number of requi-
red rounds!

Example



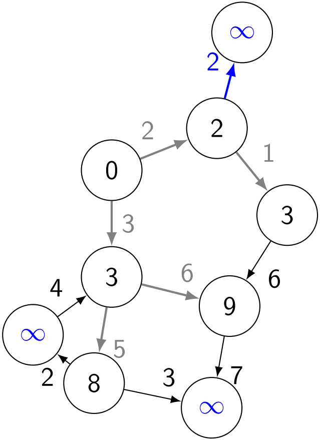
Order of edges can affects number of required rounds!

Example



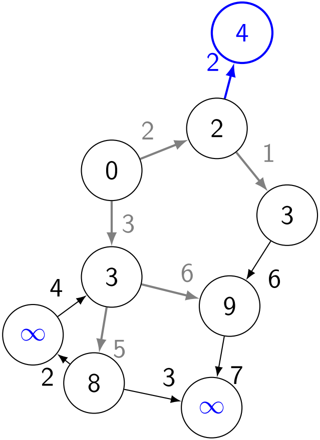
Order of edges can affects number of required rounds!

Example



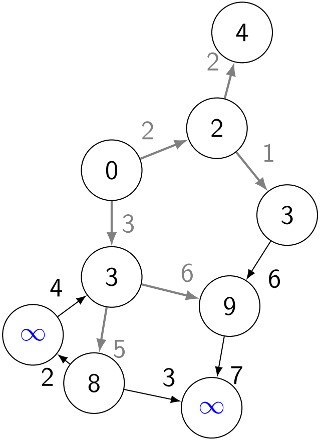
Order of edges can affects number of required rounds!

Example



Order of edges can affects number of required rounds!

Example



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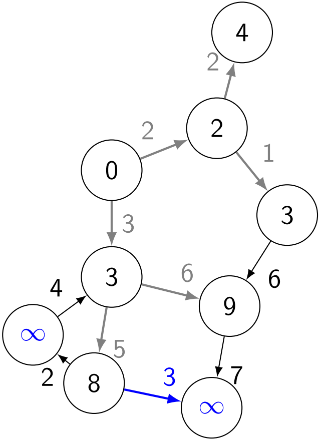
Example



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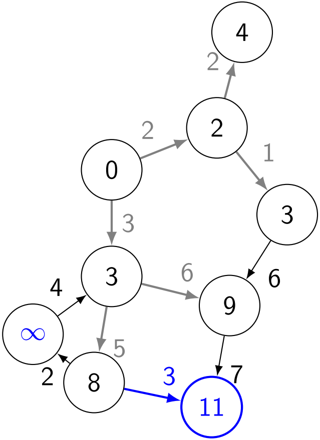

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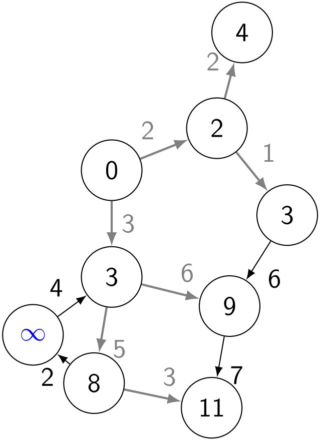
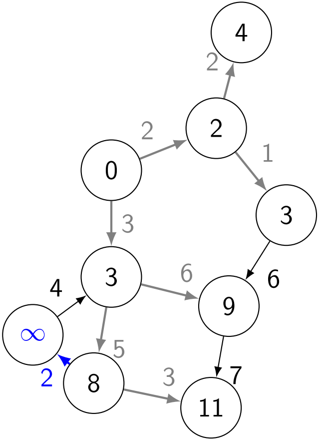
Example



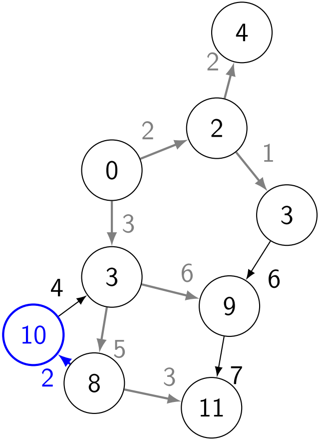
Order of edges can affects number of requi-red rounds!

Example



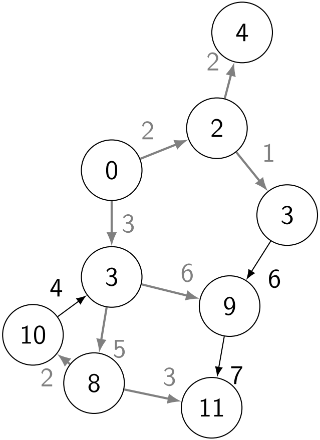
Order of edges can affects number of required rounds!

Example



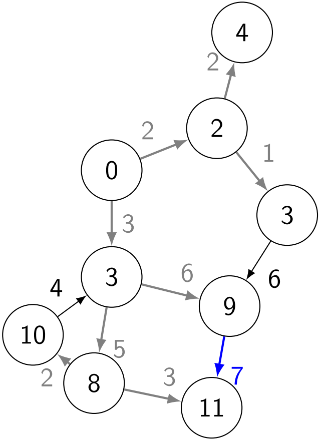
Order of edges can affects number of required rounds!

Example



Order of edges can affects number of required rounds!

Example



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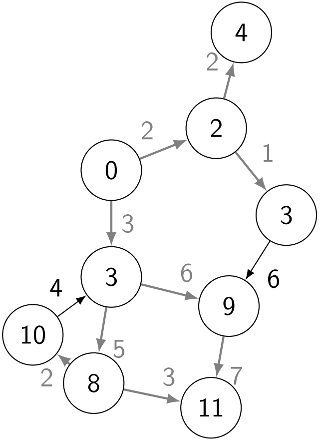

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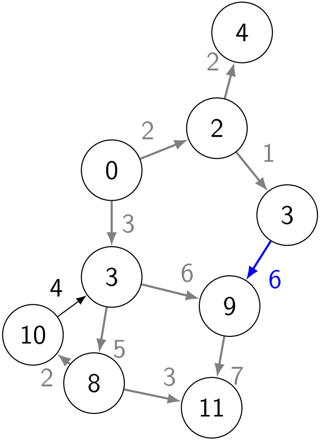

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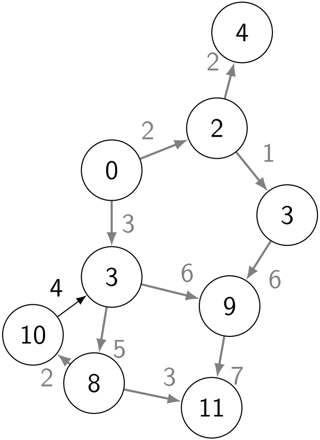

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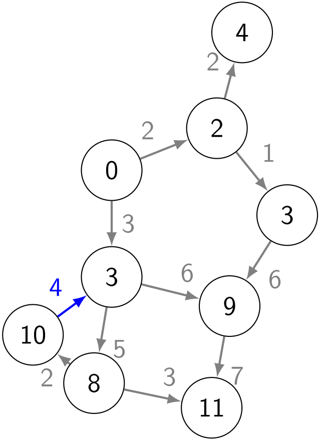
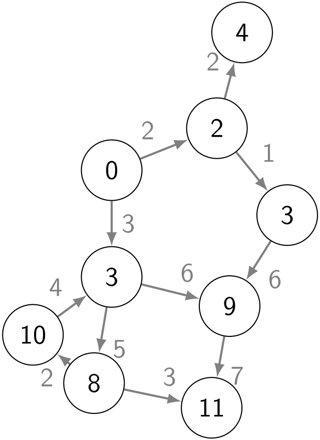


Order of edges can affects number of required rounds!

Example



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Nothing will change any more ...

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3. Assume path up to length $|V| - 1$ precise
   as no negative-weight cycles exist: any shortest path is cycle free
   thus, length at most $|V| - 1$

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- What if negative weight cycle exists?
- No shortest paths to nodes reachable from cycle!
- Bellman-Ford can detect this:
  - Iterate until D does not change.
  - If D still changed in $|V|$th round: Report negative cycle

Complexity

- In worst case, we do $|V|$ rounds
- Each round inspects $|E|$ edges
- Time for relaxing edge: $O(1)$
- Complexity is $O(|V||E|)$