COMP36111: Advanced Algorithms I

Lecture 9: Savitch's Theorem and the Immerman-Szelepcsényi Theorem

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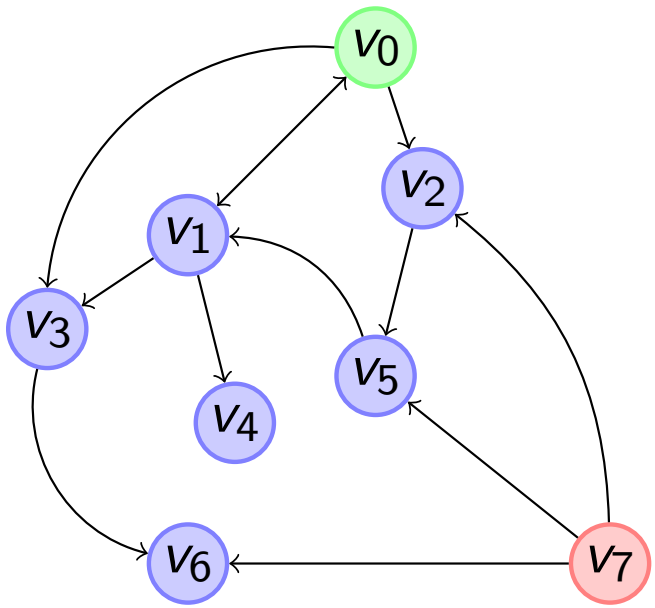
- Reading for this lecture:
  - Sipser, Ch. 8 (Space Complexity).

Outline

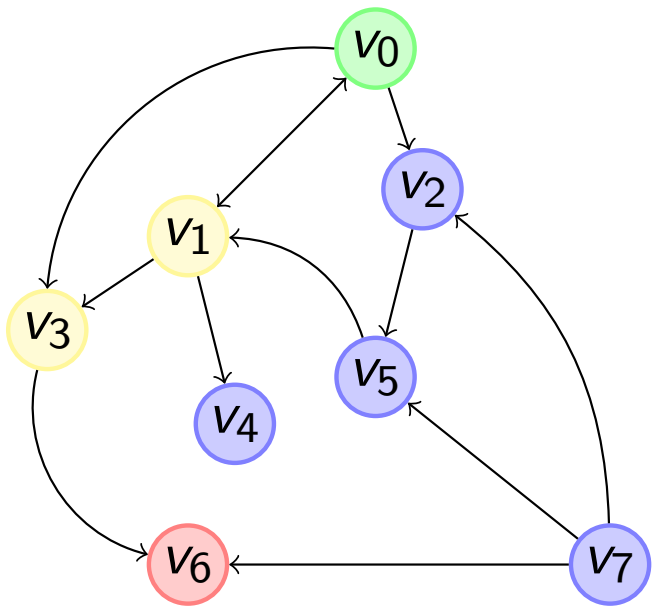
- Recall that a *directed graph* is a pair $G = (V, E)$, where V is a set and $E \subseteq \binom{V}{2}$ a set of ordered pairs of distinct elements of V.



- We assume G is encoded on the input tape of a TM as follows:

$$7 : 0, 1; 0, 2; 0, 3; 1, 0; 1, 3; 1, 4; 2, 5; 3, 6; 5, 1; 7, 2; 7, 5; 7, 6$$

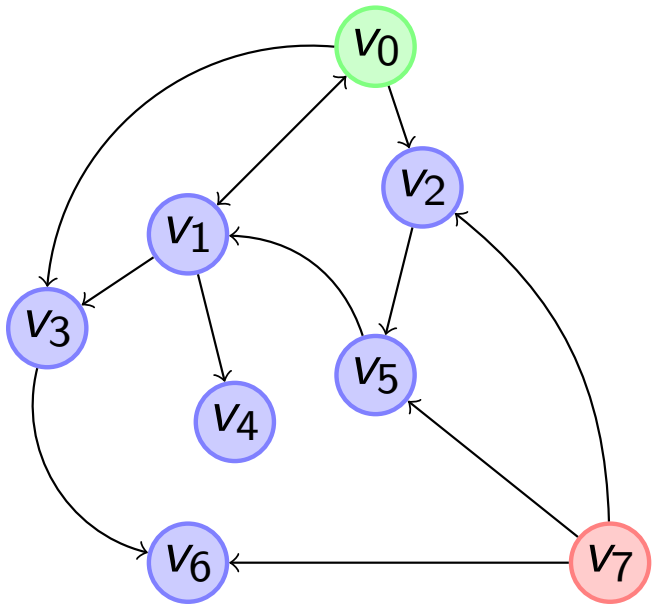
- If $G = (V, E)$ is a directed graph, and $u, v \in V$, we say that v is *reachable* from u if there exists a sequence $u = u_0, \ldots, u_m = v$ from V with $m \geq 0$ such that, for each i $(0 \leq i < m)$ $(u_i, u_{i+1}) \in E$.
- In our graph G, v_6 is reachable from v_0



- since we have the sequence

$$v_0 \to v_1 \to v_3 \to v_6.$$

- However, v_7 is not reachable from v_0.



- We then have the problem:

  REACHABILITY
  Given: A directed graph $G = (V, E)$ and nodes $s, t \in V$
  Return: Yes if t is reachable from s in G, No otherewise.

- In an earlier lecture, we gave an algorithm showing that REACHABILTY is in $\textsc{Time}(O(n))$.
- A bit of revision won't hurt ...

- An algorithm for solving REACHABILITY:

```
begin DFS-directed((V, E), u, v)
        DFS-aux((V, E), u)
        if  v is marked
          return Y
        return N
end DFS-directed

begin DFS-aux((V, E), u)
        mark u
        for each e ∈ edges(u) do
            if e = (u, w) with w unmarked do
              DFS-aux((V, E), w)
end DFS-aux
```

- We saw an essentially identical algorithm in Lecture 1a. It runs in linear time (and hence linear space).

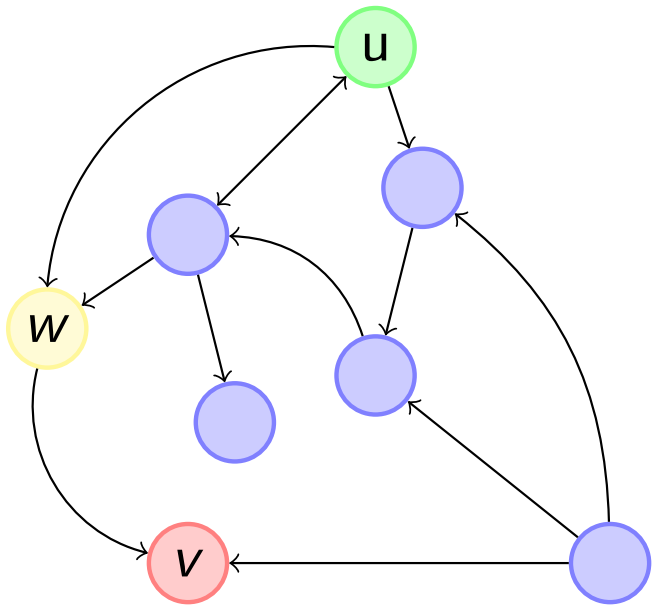
Outline

- The following deterministic algorithm outputs YES iff v is reachable from u in $G = (V, E)$ in at most 2^h steps:

```
begin isReachableNum(u,v,G,h)
    if h = 0
      if u = v or (u, v) ∈ E return Yes
      else return No
    for w ∈ V
      if (isReachableNum(u,w,G,h − 1) and
          isReachableNum(w,v,G,h − 1)) return Yes
    return No
```

- Thus, we can then solve the reachability problem by calling `isReachableNum`$(u,v,G,\lceil \log V \rceil)$

- To see how this works, we note that any path from u to v of length $\leq 2^h$ must have a midpoint w



- Hence, there is a path from u to w of length $\leq 2^{h-1}$ and a path from w to v of length $\leq 2^{h-1}$

- How do we implement this on a Turing machine?
- Answer: by keeping the triples $\langle u, v, h \rangle$ on a work-tape:

$$\langle u, v, h \rangle$$

```
begin isReachableNum(u,v,G,h)
    if h = 0
      if u = v or (u, v) ∈ E return Yes
      else return No
    for w ∈ V
      if (isReachableNum(u,w,G,h − 1) and
          isReachableNum(w,v,G,h − 1)) return Yes
    return No
```

- We see that this algorithm requires at most $O(h \cdot \log |V|)$ space.

- How do we implement this on a Turing machine?
- Answer: by keeping the triples $\langle u, v, h \rangle$ on a work-tape:

$$\langle u, v, h \rangle \langle u, w_1, h-1 \rangle$$

```
begin isReachableNum(u,v,G,h)
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- How do we implement this on a Turing machine?
- Answer: by keeping the triples $\langle u, v, h \rangle$ on a work-tape:

$$\langle u, v, h \rangle \langle u, w_1, h-1 \rangle \langle u, w_2, h-2 \rangle$$

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begin isReachableNum(u,v,G,h)
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- How do we implement this on a Turing machine?
- Answer: by keeping the triples $\langle u, v, h \rangle$ on a work-tape:

$$\langle u, v, h \rangle \langle u, w_1, h-1 \rangle \langle u, w_2, h-2 \rangle \cdots \langle u, w_\ell, h-\ell \rangle$$

```
begin isReachableNum(u,v,G,h)
    if h = 0
      if u = v or (u, v) ∈ E return Yes
      else return No
    for w ∈ V
      if (isReachableNum(u,w,G,h − 1) and
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    return No
```

- We see that this algorithm requires at most $O(h \cdot \log |V|)$ space.

- How do we implement this on a Turing machine?
- Answer: by keeping the triples $\langle u, v, h\rangle$ on a work-tape:

$$\langle u, v, h\rangle\langle u, w_1, h-1\rangle\langle u, w_2, h-2\rangle \cdots \langle w_\ell, v, h-\ell\rangle$$

begin `isReachableNum`(u,v,G,h)
  if $h = 0$
    if $u = v$ or $(u, v) \in E$ return Yes
    else return No
  for $w \in V$
    if (`isReachableNum`$(u,w,G,h-1)$ and
      `isReachableNum`$(w,v,G,h-1)$) return Yes
  return No

- We see that this algorithm requires at most $O(h \cdot \log |V|)$ space.

- Hence the call $\texttt{isReachableNum}(u, v, G, \lceil \log V \rceil)$ requires $O(\log^2 |V|)$ space.
- This proves:

Theorem (Savitch, first form)

REACHABILITY is in $\textsc{Space}(\log^2 n)$.

- Suppose we have a Turing machine M over an alphabet with c symbols, having just one tape, and running in space $f(n)$.
- By a configuration of M we mean a triple $\langle s, w, i \rangle$, where:
  - s is a state of M;
  - w is a word over the alphabet of M (tape contents);
  - $1 \leq i \leq |w|$ (head position).
- Writing $w = a_1 \ldots a_\ell$, we can conveniently encode this configuration on a second (work-) tape as

$$a_1 \ldots a_{h-1} s a_h \ldots a_\ell$$

- We can think of this as the label of a node in a graph, G.

- Consider again the configuration

$$a_1 \dots a_{h-1} s a_h \dots a_\ell$$

and suppose $s, a_h \mapsto b, \text{right}, t$ is a transition of M, where $a = a_h$.

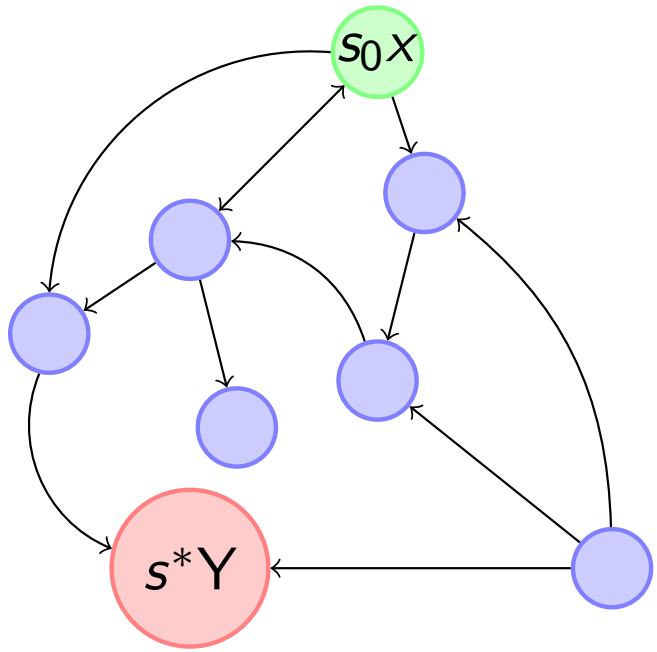
- Then we can easily compute the subsequent configuration

$$a_1 \dots a_{h-1} b t a_{h+1} \dots a_\ell$$

(on another tape if you like).

- We can think of any pair of such configurations as an edge in G

- Determining whether M has an accepting run (in time $f(n)$) now amounts to determining whether there is a path from the initial state of M to any accepting state of M



- The number of nodes of G is bounded by $S \cdot f(n) \cdot c^{f(n)}$, where S is the number of states and c the size of the alphabet;
- Note that we can compute the edges of the G on the fly: we often do not need the whole graph.

Theorem (Savitch (second form))

If f is a proper complexity function and $f(n) \geq \log n$, then $\text{NSPACE}(f) \subseteq \text{SPACE}(f^2)$.

Proof.

Suppose P is a problem in $\text{NSPACE}(f)$. Let M be a nondeterministic TM running in $\text{SPACE}(f)$, and accepting P. To determine whether $x \in P$, determine whether configuration graph of M has a path of length at most $2^{O(f(|x|))}$ from the initial node to an accepting node. This can be done in $\text{SPACE}(O(f(n))^2)$. □

Corollary

$\text{NPSPACE} = \text{PSPACE}$; $\text{NEXPSPACE} = \text{EXPSPACE}$, ...

Corollary

$\text{NPSPACE} = \text{CO-NPSPACE}$; $\text{NEXPSPACE} = \text{CO-NEXPSPACE}$, ...

Outline

- The algorithm `DFS-directed` showed that REACHABILITY is in $\textsc{Time}(n)$ and hence in $\textsc{Space}(n)$.
- The algorithm `isReachableNum` showed that REACHABILITY is in $\textsc{Space}((\log n)^2)$.
- We now present a very simple algorithm to show that, with non-determinism, we can get the space requirements down still further.

- The following non-deterministic algorithm has a run which outputs YES iff v is reachable from u in $G = (V, E)$:

```
begin isReachable(u,v,G)
    let current = u
    let counter = 0
    until current = v or counter = |V|
        guess a node u' ∈ V
        if (current, u') ∉ E return NO
        let current = u'
        increment counter
    if current = v return YES
    else return NO
```

- The only things we need to store are `current` and `counter`, which require only $\log |V|$ bits.
- Thus we see that REACHABILITY is in $\mathrm{NSPACE}(\log n)$ —i.e. $\mathrm{NLOGSPACE}$.

Theorem

REACHABILITY is NLogSpace*-complete.*

Proof.

We showed above that REACHABILITY is in NLogSpace.

Suppose L is a language recognized by a non-deterministic TM, M, running in time $O(\log n)$. Given an input x, let G be the configuration graph for M with input x. Let u be the node representing the initial configuration. We may assume this graph has a single acepting node v. Now, $x \in L$ if and only if (G, u, v) is an instance of REACHABILITY. The mapping $x \mapsto (G, u, v)$ can easily be constructed in space bounded by $\log n$. □

- It was very easy to see that REACHABILITY is in NLogSpace.
- However, let us now consider its converse:

UNREACHABILITY
Given: A directed graph $G = (V, E)$ and nodes $s, t \in V$
Return: Yes if t is not reachable from s in G, No otherewise.

- We shall now show that UNREACHABILITY is in NLogSpace too.

- Fix a directed graph $G = (V, E)$, and a node $u \in V$.
- The trick is to use a very simple non-deterministic subroutine:

```
begin reachableLossy(u,v,k)
    set u' := u
    until k = 0
        guess any node v'
        if u' ≠ v' and (u', v') ∉ E return No
        set u' := v'
        decrement k
    if u' = v return Yes
    return No
```

- `reachableLossy(`u`,`v`,`k`)` has a run returning Yes iff v is reachable from u in k or fewer steps.
- Nothing is said about runs of `reachableLossy(`u`,`v`,`k`)` returning No.

- Assume we have an algorithm `isReachableFail`(u, v, k) which, for $1 \leq k < n$, *either* returns $\perp$, Yes or No:
  - `isReachableFail` has a run returning Yes, iff v is reachable from u in at most k steps;
  - `isReachableFail` has a run returning No, iff v is not reachable from u in at most k steps;
- Then the following algorithm returns the number of nodes reachable from u in k steps or fewer, or just returns $\perp$:

```
begin numReachableFail(u, k)
    if k = 0 return 1
    set m = 0
    for i = 0, ..., n − 1
        let Q= isReachableFail(u, u_i, k)
        if Q = ⊥, then return ⊥
        if Q = Yes, then increment m
    return m
```

- Now for the definition of `isReachableFail` (assume $1 \leq k < n$):

  begin `isReachableFail`(u, v, k)
  - let $s =$ `numReachableFail`$(u, k-1)$
  - if $s = \perp$ then return $\perp$
  - let $m = 0$
  - for $i = 0, \ldots, n-1$
    - if `reachableLossy`$(u, u_i, k-1) =$ Yes
      - if $u_i = v$ or $(u_i, v) \in E$ then return Yes
      - increment m
  - if $m < s$ then return $\perp$
  - return No

- Now for the our non-deterministic algorithm accepting UNREACHABILITY:

  begin isUnreachable(u, v, (V, E))
  if `isReachableFail`(u, v, $|V| - 1$) = No then return Yes
  return No

- It is easy to see that this algorithm requires only logarithmic space, and has a run returning Yes if and only if v is not reachable from u in $G = (V, E)$.

Theorem (Immerman-Szelepcsényi, first form)

UNREACHABILITY *is in* NLogSpace.

Theorem (Immerman-Szelepcsényi (second form))

If f is a proper complexity function and $f(n) \geq \log(n)$, then $\text{NSPACE}(f) = \text{CO-NSPACE}(f)$.

Proof.

Suppose P is a problem in $\text{NSPACE}(f)$, and let $\bar{P}$ be its complement problem. Let M be a nondeterministic TM running in $\text{SPACE}(f)$, and accepting P, and let x be an input string. Denote by G be the configuration graph of M with input x. Then x is a positive instance of $\bar{P}$ if and only if the node of G representing a successful run is unreachable from the start node of G. □

Corollary

$\text{NLOGSPACE} = \text{CO-NLOGSPACE}$.

Corollary

KROM-SAT (= 2-SAT) is in NLOGSPACE.

Theorem

The problem KROM-SAT (= 2-SAT) is NLogSpace*-complete.*

Proof.

We have just shown that KROM-SAT is in NLogSpace.
For NLogSpace-hardness, we reduce the problem UNREACHABILITY (=Co-REACHABILITY) to KROM-SAT. Let $G = (V, E)$ be a directed graph, and $u_0, v_0 \in V$ be vertices. Treating the vertices V as propositional variables, define the set of clauses Γ_G

$$\{u_0, \neg v_0\} \cup \{u \to v \mid (u, v) \in E\}.$$

□

Proof.

If $u_0, \ldots, u_m = v_0$ is a path through G, then any truth-value assignment marking u_0 and $\{u \to v \mid (u, v) \in E\}$ true must make v_0 true. Hence Γ_G is unsatisfiable.

Conversely, if v_0 is not reachable from u_0, let θ be the truth-value assignment

$$\theta(v) = \begin{cases} T & \text{if } v \text{ is reachable from } u_0 \text{ in } G; \\ F & \text{otherwise.} \end{cases}$$

Then θ evidently satisfies Γ_G.

NLogSpace-hardness of KROM-SAT then follows from the NLogSpace-hardness of UNREACHABILITY. □

Outline

- Consider again the classes $\textsc{Time}(f)$, $\textsc{NTime}(f)$, $\textsc{Space}(f)$.
- How are these related?
- First of all, it is trivial that

$$\textsc{Time}(f) \subseteq \textsc{NTime}(f).$$

- What about $\textsc{NTime}(f)$ and $\textsc{Space}(f)$?
  - Consider a TM, M, running in $\textsc{NTime}(f)$, and acting on input x of length n.
  - M can make at most $f(n)$ non-deterministic choices.
  - Represent these choices as a digits as a string s of length $f(n)$.
  - We can run through all possible strings, guiding the choices of M using s, succeeding if M ever succeeds.
  - The resulting deterministic TM takes no more space, and recognizes the same language as M.
- Thus, we have

$$\textsc{NTime}(f) \subseteq \textsc{Space}(f).$$

- To relate $\textsc{Space}(f)$ to a time-complexity class, consider a TM, M, running in $\textsc{Space}(f)$, and operating on an input x of length n.
- We may suppose M has K tapes ($K-2$ work-tapes) and uses an alphabet Σ.
- A configuration is given by $K-2$ strings of length at most $f(n)$, together with one of $|Q|$ states, and a K-tuple of integers ($\leq f(n)$) indicating the head position on each tape.
- The total number of such configurations is $|Q|.|\Sigma|^{(K-2)f(n)+K\log(f(n))}$, and reachability in such a graph can be decided in time $c' \cdot |Q|.|\Sigma|^{(K-2)f(n)+K\log(f(n))}$ for some constant c'.
- Thus, we have

$$\textsc{Space}(f(n)) \subseteq \cup_{a>0}\textsc{Time}(2^{af(n)}).$$

- Applying these results to the larger classes PTIME, NPTIME, PSPACE etc, a nice picture emerges:

$$\text{PTIME} \subseteq \text{NPTIME} \subseteq \text{PSPACE} \subseteq \text{EXPTIME} \ldots$$

- What about the non-deterministic stuff?
- We know from Savitch's theorem that $\mathrm{PSpace} = \mathrm{NPSpace}$, $\mathrm{ExpSpace} = \mathrm{NExpSpace}$ etc., so we can henceforth ignore large non-deterministic space-classes.
- Warning: it does not follow from Savitch's theorem that $\mathrm{LogSpace} = \mathrm{NLogSpace}$, and we do not know whether this equation holds.
- Warning: it does not follow from Savitch's theorem that $\mathrm{PTime} = \mathrm{NPTime}$, etc, and we do not know whether these equations hold.

- Thus, we have:

$$\textsc{LogSpace} \subseteq \textsc{NLogSpace} \subseteq \textsc{PTime} \subseteq \textsc{NPTime} \subseteq \textsc{PSpace} \subseteq \textsc{ExpTime} \subseteq \textsc{NExpTime} \subseteq \textsc{ExpSpace} \subseteq \cdots$$

- We showed earlier that $\textsc{PTime} \subsetneq \textsc{ExpTime}$, and hence that at least one of the inequalities

$$\textsc{PTime} \subseteq \textsc{NPTime}, \quad \textsc{NPTime} \subseteq \textsc{PSpace}, \quad \textsc{PSpace} \subseteq \textsc{ExpTime}$$

is strict; but it is not know which.

- How do the complements of these classes fit into the picture?
- We have already estabiashed the following:
  - deterministic classes (time or space) are always equal to their complement classes;
  - non-deterministic space classes from NPSpace upwards are equal to their deterministic variants (Savitch) and hence to their complement classes;
  - NLogSpace is equial to its complement class (Immerman-Szelepcseńyi).
- We do not know whether common non-deterministic time classes, such as NPTime, NExpTime etc., are equal to their complements.