COMP36111: Advanced Algorithms I
Lecture 7: Hardness and Reductions

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- Reading for this lecture:
  - Sipser: Chapter 7.

Outline

Reductions

- Recall the problems SAT and k-SAT

SAT

Given: A set of clauses Γ
Return: Y if Γ is satisfiable, and N otherwise

k-SAT

Given: A set of clauses Γ each of which has at most k literals.
Return: Y if Γ is satisfiable, and N otherwise.

- *Prima facie*, SAT looks harder than k-SAT. But is it?

- Let P_1, P_2 be problems over alphabets Σ_1, Σ_2, respectively.
- We say P_1 is (*many-one logspace*) *reducible* to P_2 if there is a function $f : \Sigma_1^* \to \Sigma_2^*$ such that: (i) f can be computed by a deterministic TM using at most $\log n$ space on any work tape; and (ii) for all $x \in \Sigma_1^*$, $x \in P_1$ if and only if $f(x) \in P_2$.
- In this case, we write

$$P_1 \leq_m^{\log} P_2$$

- We think of $P_1 \leq_m^{\log} P_2$ as stating any of the following:
  - P_2 is at least as hard as P_1;
  - P_1 is no harder than P_2;
  - if anyone shows me an easy way of solving P_2, I have an easy way of solving P_1.

- Such reductions provide a way of showing that a problem is in a complexity class, because (sensible) complexity classes, such as

$$\textsc{LogSpace}, \textsc{NLogSpace}, \textsc{PTime}, \textsc{NPTime}, \ldots$$

  are closed under many-one logspace reductions.
- Warning: Classes such as $\textsc{Time}(n)$, $\textsc{Time}(n^2)$ etc. are not closed under many-one logspace reductions.

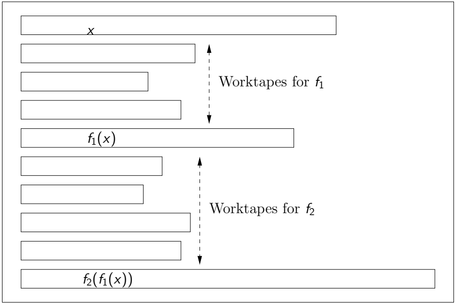
Outline

Furthermore, reducibility is a transitive relation, as the next theorem shows.

Theorem

If $f_1 : \Sigma_1^* \to \Sigma_2^*$ *and* $f_2 : \Sigma_2^* \to \Sigma_3^*$ *are both computable in logaithmic space, then so is* $f_2 \circ f_1 : \Sigma_1^* \to \Sigma_3^*$.

The following picture is not a proof!



- Here is a Turing machine that will compute $f_2 \circ f_1$ in logarithmic space:

  calculate the first bit of $f_1(x)$

  keep a counter to say which bit this is—initially 1

  start a simulation of $f_2(f_1(x))$, using the calculated bit

  if the simulation of f_2 asks to move the read head to the right

      calculate next bit of $f_1(x)$

      write it on top of the current bit

      update the output bit counter

  if the simulation of f_2 asks to move the read head to the left

      restart the calculation of $f_1(x)$

      continue until the required output bit is calculated

      write it on top of the current bit

      update the output bit counter

- A weaker notion of reduction is commonly encountered in textbooks (e.g. Sipser).
- Denote by **P** the set of functions $\{n^c \mid c > 0\}$.
- Let P_1, P_2 be problems over alphabets Σ_1, Σ_2, respectively.
- We say P_1 is (*many-one polytime*) *reducible* to P_2 if there is a function $f : \Sigma_1^* \to \Sigma_2^*$, in $\textsc{Time}(\mathbf{P})$ such that, for all $x \in \Sigma_1^*$, $x \in P_1$ if and only if $f(x) \in P_2$.
- In this case, we write

$$P_1 \leq_m^p P_2$$

- Many-one logspace reducibility is at least as strong as many-one polytime reducibility.
- Many-one polytime reducibility is obviously transitive. (Ask if you do not understand this.)
- However, many-one logspace reducibility is theoretically a bit more useful.
- In practice, most encountered instances of many-one polytime reducibility are in fact instances of many-one logspace reducibility.
- We shall always use many-one logspace reducibility unless explicitly stated otherwise.

Outline

- It turns out that, for certain complexity classes $\mathcal{C}$, and certain problems P, *every* problem $P' \in \mathcal{C}$ is reducible to P.
- That is, P is at least as hard as every problem in $\mathcal{C}$.
- Of particular interest is where the problem P is itself a member of $\mathcal{C}$.
- Much of the attraction of complexity theory arises from the existence of such problems.

Definition

Let $\mathcal{C}$ be a complexity class and P a problem. We say that P is $\mathcal{C}$-*hard* (*under many-one logspace reducibility*) if, for all $P' \in \mathcal{C}$, $P' \leq_m^{\log} P$.

We say that P is $\mathcal{C}$-*complete* (umolsr) if, $P \in \mathcal{C}$ and P is $\mathcal{C}$-*hard* (umolsr).

Outline

Theorem (Cook)

SAT is NPTIME*-complete.*

Proof.

Suppose $\mathcal{P}$ is any problem in $\mathrm{NPT{\small IME}}$. Let M be a TM accepting $\mathcal{P}$, with running time bounded by $p(n)$. For simplicity, let us assume M has just one tape. Thus, M has the form

$$\langle \Sigma, Q, s^*, T \rangle,$$

where Σ is the alphabet of $\mathcal{P}$, Q is the set of states, s^* the halting state and T the set of transitions.

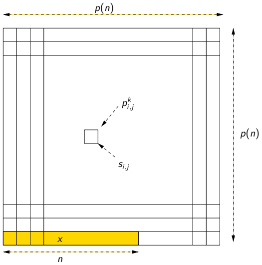
Each transition $\tau \in T$ has the form

$$\tau = \langle s, a, t, b, \delta \rangle,$$

where $s, t \in Q$ are states, $a, b \in \Sigma \cup \{\sqcup, \triangleright\}$, and $\delta \in \{-1, 0, 1\}$ indicating 'left', 'stay' or 'right'. □

Proof.

We picture the operation of M as



and encode any run using the proposition letters

$p^a_{i,j}$: tape square i contains symbol a at time j
$h_{i,j}$: the head is over tape square i at time j
q^s_j: the state is s at time j.
$t^\tau_{i,j}$: transition τ is executed at time j with head on tape square i.

Proof.

We write clauses saying that, at each time, the head is somewhere

$$\{h_{1,j} \vee \cdots \vee h_{p(n),j} \mid 1 \leq j \leq p(n)\}$$

and is not in two places at once

$$\{\neg h_{i,j} \vee \neg h_{i',j} \mid 1 \leq i < i' \leq p(n), 1 \leq j \leq p(n)\}$$

and so on. We write clauses saying that the input is $x[1], \ldots, x[n]$ (remember $\sqcup$ is the blank symbol):

$$\{p_{i,1}^{x[i]} \mid 1 \leq i \leq n\}$$
$$\{p_{i,1}^{\sqcup} \mid n+1 \leq i \leq p(n)\}$$

and so on. (proof TBC ...) □

Proof.

Further, we write clauses specifying when a transition of M may be executed. For all i, j $(1 \leq i, j \leq p(n))$, and for all $a \in \Sigma \cup \{\sqcup, \triangleright\}$, we take Γ_x to contain the (big) clause

$$\neg q_j^s \vee \neg h_{i,j} \vee \neg p_{i,j}^a \vee \bigvee \{ t_{i,j}^\tau \mid \tau = \langle s, a, t, b, \delta \rangle \in T \}$$

listing the allowed transitions M may make. Note that M is a non-deterministic TM! □

Proof.

And we write clauses specifying the effects of transitions:

$$\{\neg t^{\tau}_{i,j} \lor p^{b}_{i,j+1} \mid 1 \leq i,j \leq p(n),\ \tau = \langle s,a,t,b,\delta\rangle\}$$
$$\{\neg t^{\tau}_{i,j} \lor q^{t}_{j+1} \mid 1 \leq i,j \leq p(n),\ \tau = \langle s,a,t,b,\delta\rangle\}$$
$$\{\neg t^{\tau}_{i,j} \lor h_{i+\delta,j+1} \mid 1 \leq i,j \leq p(n),\ \tau = \langle s,a,t,b,\delta\rangle\}.$$

Actually, there are some complications here when the tape head is over the leftmost square. Can you fix this formula? □

Proof.

And we write clauses saying that M accepts the input:

$$\{q^{s^*}_{p(n)}, p^{\mathrm{Y}}_{1,p(n)}\} \cup \{p^{\sqcup}_{i,p(n)} \mid 2 \leq i \leq p(n)\},$$

where s^* is the halting state.

Call the resulting set of clauses Γ_x.
There are a few additional clauses in Γ_x that I have not mentioned; but it is routine to fill them in. (proof TBC . . .) □

Proof.

It is easy to see that Γ_x is satisfiable iff M accepts x; hence Γ_x is satisfiable iff $x \in P$.

It is also 'easy' to see that, from a description of x, we can compute the set of clauses Γ_M using at most $\log n$ amount of workspace, where $n = |x|$. (Remember: the parameters of M are constant here; the only variable input is x.)

Thus, the function $x \mapsto \Gamma_x$ shows that $P \leq_m^{\log}$ SAT, as required. □

- It is completely trivial that 3-SAT is no harder than SAT.
- Slightly surprising is that the reverse condition holds: SAT is no harder than 3-SAT!
- Notice that this means that 3-SAT is $\mathrm{NPT}\textsc{ime}$-complete.
- For suppose $\mathcal{P}$ is a problem in $\mathrm{NPT}\textsc{ime}$. We have

$$\mathcal{P} \leq_m^{\log} \mathsf{SAT} \leq_m^{\log} \text{3-}\mathsf{SAT}$$

and the result follows by the transitivity of $\leq_m^{\log}$.

Outline

Theorem

3-SAT is NPTIME*-complete*

Proof.

We show that SAT $\leq_m^{\log}$ 3-SAT.

Suppose we are given a set of clauses Γ. We show how to compute a set of 3-literal clauses Γ' such that Γ is satisfiable iff Γ' is satisfiable.

Pick any $(\ell_1 \vee \cdots \vee \ell_m) \in \Gamma$ with $m \geq 4$. (proof TBC ...)

□

Proof.

Let p be a new proposition letter, and let Γ'' be the result of replacing γ in Γ with the pair of clauses:

$$p \vee \ell_3 \vee \cdots \vee \ell_m$$
$$\neg p \vee \ell_1 \vee \ell_2$$

These clauses entail γ, so if Γ'' is satisfiable, Γ certainly is. On the other hand, if the assignment θ satisfies Γ, then setting $\theta(p) = \theta(\ell_1 \vee \ell_2)$ clearly satisfies Γ''.

Proceeding in this way, we eventually obtain the required Γ'. □

Outline

Reductions and hardness

Reductions

Transitivity of reductions

Hardness and completeness

Cook's theorem

Cook's theorem

Some easy reductions

3-SAT

Integer linear programming

- Integer linear programming (ILP) is the problem of determining the existence of a solution (over $\mathbb{N}$) to a system of linear Diophantine equations.

ILP
Given: a system of l.d. equations $\mathcal{E} : A\mathbf{x} = \mathbf{b}$.
Return: Yes if $\mathcal{E}$ has a solution over $\mathbb{N}$, and No otherwise.

- We are also interested in the special case where the solutions are limited to values 0 and 1
- For $k \geq 2$, we have the problem

ILP(0/1)
Given: a system of l.d. equations $\mathcal{E} : A\mathbf{x} = \mathbf{b}$.
Return: Yes if $\mathcal{E}$ has a solution over $\{0, 1\}$, and No otherwise.

Theorem

ILP(0/1) is NPTIME*-complete*

Proof.

We show that 3-SAT $\leq_m^{\log}$ ILP(0/1).

Suppose we are given a set of 3-literal clauses Γ. We show how to compute system of linear Diophantine equations $\mathcal{E}$ such that $\mathcal{E}$ has a solution over $\{0, 1\}$ iff Γ is satisfiable.

For every proposition letter p mentioned in Γ, let x_p and $x_{\neg p}$ be variables and write the equation

$$x_p + x_{\neg p} = 1.$$

. □

Proof.

For every clause $\gamma := (\ell_1 \vee \ell_2 \vee \ell_3) \in \Gamma$, let y_1^γ, y_2^γ be variables, and write the equation

$$x_{\ell_1} + x_{\ell_2} + x_{\ell_3} + y_1^\gamma + y_2^\gamma = 3.$$

Call the resulting system of equations $\mathcal{E}_\Gamma$.

Suppose θ is a truth-value assignment for the proposition letters in Γ. Now define

$$x_p = \begin{cases} 1 & \text{if } \theta(p) = \top \\ 0 & \text{otherwise.} \end{cases}$$

and define $x_{\neg p} = 1 - x_p$.

□

Proof.

If θ makes $\gamma := (\ell_1 \vee \ell_2 \vee \ell_3)$ true, then we can certainly find y_1^γ, y_2^γ satisfying.

$$x_{\ell_1} + x_{\ell_2} + x_{\ell_3} + y_1^\gamma + y_2^\gamma = 3.$$

So all the equations in $\mathcal{E}_\Gamma$ are satisfied.

Conversely, given any assignment of values in $\{0, 1\}$ to the variables x_ℓ and y_j^γ, define the truth-value assignment

$$\theta(p) = \begin{cases} \top & \text{if } x_p = 1 \\ \bot & \text{otherwise.} \end{cases}$$

If the various equations $x_p + x_{\neg p} = 1$ hold, then, for all literals ℓ, $\theta(p) = \top$ iff $x_\ell = 1$. Hence, if the remaining equations in $\mathcal{E}_\Gamma$ hold, every clause in Γ is made true by θ. □