Special instructions: You may work either individually or in a group of exactly two persons. Write your solutions out on paper and deliver them (with one photocopy thereof) to SSO by 14:00 on Thursday, 27th November, 2014. If you are working in a group, one script should be submitted for the group, with both names at the top; both members of the group will then receive the same mark. Clearly write your name(s), student ID number(s) and the words "Comp36111 Sec. B Coursework" on the front (cover) sheet and staple all sheets together. Submitted solutions must be entirely the effort of the persons whose names appear at the top of the submission.

UNIVERSITY OF MANCHESTER
SCHOOL OF COMPUTER SCIENCE

Advanced Algorithms I: Coursework for Sec. B

Time: This should take you about six hours

Please answer all questions.

Marks will be awarded for clarity and succinctness as well as correctness.

The use of electronic calculators is not recommended.

Section B

1. Recall that a *multiset* is a set in which elements may have multiple occurrences. Thus, $\{1,2,4,1,2,5,0,5\}$ is a multiset. Note that elements in a multiset have no order. Thus, $\{1,2,4,1,2,5,0,5\} = \{1,2,4,1,2,5,5,0\}$. A *partition* of a multiset S is a collection of multisets $S_1,\ldots,S_m$ such that no S_k $(1 \leq k \leq m)$ contains any element not occurring in S, and the total number of occurrences of any element x in S equals the sum of the numbers of its occurrences in the S_k. (We allow the S_k to be empty.) Thus, for example, $S = \{1,2,4,1,2,5,0,5\}$ can be partitioned into the two multisets $S_1 = \{1,2\}$ and $S_2 = \{1,2,4,5,0,5\}$. If S is a multiset of numbers, then the *sum* of S, written $\sum S$, is simply the result of adding up all the numbers in question. Thus, $\sum\{1,2,4,1,2,5,0,5\} = 20$.

   Consider the following problem.

   > 2-NO-MAJORITY
   > Given: a multiset S of positive integers.
   > Return: Y if S can be partitioned into multisets S_1 and S_2 such that $\sum S_1 = \sum S_2$;
   > N otherwise.

   The name makes more sense if one expresses the operative condition on S_1 and S_2 by saying that neither S_1 nor S_2 should constitute a (weighted) majority of S.

   Consider also the following problem.

   > SUBSETSUM
   > Given: a multiset S of positive integers and a positive integer $b < \sum S$.
   > Return: Y if there exists a sub-multiset $S' \subseteq S$ such that $\sum S' = b$;
   > N otherwise.

   We will show in class that SUBSETSUM is NPTIME-hard.

   a) Show that 2-NO-MAJORITY is in NPTIME. (2 marks)

   b) Let an instance (S,b) of SUBSETSUM be given. Define the multiset

   $$S^* = S \cup \{(2\sum S - b), (\sum S + b)\}.$$

   Show that, if S^* is partitioned into multisets S_1^*, S_2^* such that $\sum S_1^* = \sum S_2^*$, then S_1^* (and hence also S_2^*) contains exactly one of the entries $(2\sum S - b)$ or $(\sum S + b)$. (2 marks)

   c) Suppose that S_1^* in fact contains $2\sum S - b$, and define

   $$S' = S_1^* \setminus \{2\sum S - b\},$$

   so that S' is a sub-multiset of S. Show that $\sum S' = b$. (2 marks)

d) Suppose conversely that S has a sub-multiset S'' such that $\sum S'' = b$. Define a partition of S^* into multisets whose sums are equal. Justify your answer.

(2 marks)

e) Show that the mapping $(S, b) \mapsto S^*$ defined above is a reduction, and deduce that 2-NO-MAJORITY is NPTIME-hard. (2 marks)

2. Consider the following problems, for all $m \geq 3$.

> <u>m-NO-MAJORITY</u>
> Given: a multiset S of positive integers.
> Return: Y if S can be partitioned into multisets $S_1, \ldots, S_m$ such that, for all k ($1 \leq k \leq m$), $\sum S_k \leq \sum_{1 \leq j \leq m}^{j \neq k} \sum S_j$; N otherwise.

One might express the operative condition on $S_1, \ldots, S_m$ by saying that no element of the partition should constitute a (weighted) majority of S.

a) It is obvious that if S contains any single element greater than the sum of all the others, then the answer to this problem is N. Show that, when $m \geq 3$, the reverse implication in fact obtains. That is: shows that, if S contains no single element greater than the sum of all the others, then the answer to this problem is Y.

[Hint: it suffices to show the result for $m = 3$; list the numbers in S (in any order), and start adding them up from the left; now stop just before you get a majority.]

(4 marks)

b) Hence show that, for $m \geq 3$, m-NO-MAJORITY is in LOGSPACE. You should give a careful proof, using pseudo-code, to describe key parts of your algorithm. [Hint, you have to take each of the numbers and show that it is not greater than the sum of all the others. But you do not have room to add up all the other numbers and store the result; you will have to do the comparison bit-by-bit.] (6 marks)

END OF EXAMINATION