*Special instructions: You may work either individually or in a group of exactly two persons. Write your solutions out on paper and deliver them to SSO by 15:00 on Friday, 29th November, 2013. If you are working in a group, one script should be submitted for the group, with both names at the top; both members of the group will then receive the same mark. Clearly write your name(s), student ID number(s) and the words "Comp36111 Sec. B Coursework" on the front (cover) sheet and staple all sheets together. Submitted solutions must be entirely the effort of the persons whose names appear at the top of the submission.*

**UNIVERSITY OF MANCHESTER**
**SCHOOL OF COMPUTER SCIENCE**

# Advanced Algorithms I: Coursework for Sec. B

**Time: This should take you a few hours**

Please answer all questions.

Marks will be awarded for clarity and succinctness as well as correctness.

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The use of electronic calculators is not recommended.

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## Section B

1. Consider the system of linear equations $A\mathbf{x} = \mathbf{b}$ with all coefficients in $\mathbb{N}$. We can write this explicitly as

$$\begin{array}{rcl} a_{1,1}x_1 + \cdots + a_{1,L}x_L & = & b_1 \\ \vdots \qquad\qquad \vdots & & \vdots \\ a_{m,1}x_1 + \cdots + a_{m,L}x_L & = & b_m. \end{array}$$

You should think of $L$ as a very large number, and $m$ as a rather smaller number. That is, we have few constraints in many variables.

If $I$ is any subset of the indices $\{1,\ldots,L\}$, we denote by $A_I$ the result of deleting all columns of $A$ other than those indexed by $I$, and we denote by $\mathbf{x}_I$ the result of deleting all entries of $\mathbf{x}$ other than those indexed by $I$. Thus, the expression $A_I\mathbf{x}_I$ makes sense: it is a list of $m$ linear terms in $I$ variables. Check you understand this.

Suppose that $\mathbf{u} = (u_1,\ldots,u_L)$ is a solution of $A\mathbf{x} = \mathbf{b}$ over $\mathbb{N}$. Let $K = \{\ell \mid 1 \leq \ell \leq L,\ u_\ell > 0\}$, and let $k = |K|$. Thus, $K$ is the set of indices for which $\mathbf{u}$ has a non-zero entry, and $k$ is the number of non-zero entries in $\mathbf{u}$. Suppose in addition that $k$ is *minimal*: there are no solutions over $\mathbb{N}$ with fewer than $k$ non-zero entries.

a) Let $I$ and $J$ be subsets of $K$. Show that, if $I \neq \emptyset$, but $I \cap J = \emptyset$, then the vectors $A_I\mathbf{u}_I$ and $A_J\mathbf{u}_J$ must be different. (Hint: suppose they are the same, and show that $k$ is non-minimal!) (6 marks)

b) Show that, if $I \neq J$, then the vectors $A_I\mathbf{u}_I$ and $A_J\mathbf{u}_J$ must be different. (3 marks)

c) Let $b = \max_{i=1}^{m} b_i$. Show that $k \leq m\log_2(b+1)$. (6 marks)

We will use the above result to derive an interesting complexity-theoretic result in the lectures.

2. That's interesting: if there is a solution of $A\mathbf{x} = \mathbf{b}$ over $\mathbb{N}$, then there is a solution over $\mathbb{N}$ in which at most $m\log_2(b+1)$ values are non-zero, where $m$ is the number of equations in the system, and $b$ is the maximum entry in $\mathbf{b}$.

a) Suppose we seek solutions over $\mathbb{Q}$, rather than $\mathbb{N}$. For what number $k$ can we guarantee that, if if there is a solution of $A\mathbf{x} = \mathbf{b}$ over $\mathbb{Q}$, then there is a solution over $\mathbb{Q}$ in which at most $k$ values are non-zero? State the theorem clearly and precisely. The intention is that you use books and find this out; don't try and derive the result yourself—it will take too much time if you don't do it just the right way. (8 marks)

b) Fix $m \geq 6$. Let $A$ be the $m \times (m+1)$-matrix given by

$$A = \left(\begin{array}{ccccccc|cccccc}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & \ldots & & & & 0 \\
0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & \ldots & & & & 0 \\
0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & \ldots & & & & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & \ldots & & & & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & \ldots & & & & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & \ldots & & & & 0 \\
\vdots & & & & & & & & & & & & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 1 & 1 & 1 \\
\hline
1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & \ldots & & & & 0
\end{array}\right),$$

in which a pattern of three 1s is shifted right across the first $(m-1)$ rows, and the last row contains the seven entries shown on the left followed by $(m-6)$ 0s. Let $\mathbf{b}$ be the column vector of length $m$ given by

$$\mathbf{b} = (3, 3, \ldots, 3, 4)^T$$

consisting of $(m-1)$ 3s and a single 4. Show that the unique solution of the system of Boolean equations $A\mathbf{x} = \mathbf{b}$ over $\mathbb{N}$ is the column vector $(1, \ldots, 1)^T$ consisting of $(m+1)$ 1s. How does this result compare with your answer to Part (a) of this question? (7 marks)



END OF EXAMINATION