Comp 36111 Coursework B 2013-14

Ian Pratt-Hartmann 29. 11. 13

1 a) Suppose, for contradiction that $I \neq \emptyset$, $I \cap J = \emptyset$ and $A_I \underline{u}_I = A_J \underline{u}_J$.

Let $\underline{u}'$ be the same as $\underline{u}$ except that entries indexed by ~~I~~ J are doubled, and entries indexed by I are zeroed:

$$\underline{u}'[i] = \begin{cases} 2\,\underline{u}[i] & \text{if } i \in J \\ 0 & \text{if } i \in I \\ \underline{u}[i] & \text{ow} \end{cases}$$

Since $I \cap J = \emptyset$, this is well-defined.

Then
$$A \underline{u}' = A\underline{u} + A_I \underline{u}_J - A_J \underline{u}_I$$
$$= A\underline{u} = \underline{b}$$

So $\underline{u}'$ is a solution. But since $I \neq \emptyset$, k is not minimal, a contradiction.

1 b) Suppose $I \neq J$. Let $I' = I \setminus J$, $J' = J \setminus I$ and $H = I \cap J$. Since $I \neq J$ either I' or J' is nonempty. By interchanging I and J if necessary, assume $I' \neq \emptyset$. Obviously $I' \cap J' = \emptyset$

Since
$$A_I \underline{u}_I = A_{I'} \underline{u}_{I'} + A_H \underline{u}_H$$
$$A_J \underline{u}_J = A_{J'} \underline{u}_{J'} + A_H \underline{u}_H$$

we have $A_I \underline{u}_I = A_J \underline{u}_J$ iff $A_{I'} \underline{u}_{I'} = A_{J'} \underline{u}_{J'}$.
But $A_{I'} \underline{u}_{I'} \neq A_{J'} \underline{u}_{J'}$ by part a).

1c) The mapping $I \to A_I \underline{y}_I$ is 1-1 by part b).
The number of values of $I \subseteq K$ is 2^k.
The number of values of $A_I \underline{y}_I$ is $\leq (b+1)^m$, since there are m entries taking the possible values $0, \ldots b$. Hence

$$2^k \leq (b+1)^m$$

$$\therefore k \leq m \log_2 (b+1).$$

2a) There are various ways to do this. The following fact is much stronger than we need.

Carathéodory's theorem: If $\underline{a}_1, \ldots \underline{a}_z, \underline{b}$ are column vectors of dimension (length) m, and there are non-negative ~~positive~~ rationals $\lambda_1, \ldots \lambda_z$ s.t. $\lambda_1 \underline{a}_1 + \ldots \lambda_z \underline{a}_z = \underline{b}$, then there is a selection ~~are~~ ~~a subset~~ $i(1) \ldots i(m)$ of just m of ~~these~~ these indices and non-negative rationals ~~s.t.~~ $\mu_1, \ldots \mu_m$ such that

$$\mu_1 \underline{a}_{i(1)} + \ldots + \mu_m \underline{a}_{i(m)} = \underline{b}$$

This states that if $A\underline{x} = \underline{b}$ has a solution over the non-negative rationals if and only if it has a solution over the non-negative rationals in which at most m values are non-zero.

In this question, we are interested in the solution over $\mathbb{Q}$. That's no problem, rewrite the equations as $A\underline{x}' + (-A)\underline{x}'' = \underline{b}$, get a solution over the non-negative rationals with at most m non-zero values and write $x = x' - x''$

2b Suppose $\underline{b} = (b_1, \dots b_{m+1})^T$ is a solution of $A\underline{x} = \underline{c}$ over $\mathbb{N}$. From the first equation $b_1 + b_2 + b_3 = 3$ whence $b_1\ b_2\ b_3$ are (i) the integers 0, 0, 3 in some order, or (ii) the integers 0, 1, 2 in some order, or (iii) the integers 1, 1, 1. From equations 2 to $m-1$, it is easy to see that these values must recur, in the same order, to the end of $\underline{b}$.

Thus, $\underline{b} = (b_1, b_2, b_3, b_1, b_2, b_3, b_1 \dots)^T$

From the last equation $3b_1 + b_2 = 4$

if $b_1 = 0$, ~~b_2~~ since $b_2 \leq 3$ we have $4 = 3b_1 + b_2 \leq 3$ a contradiction. Hence $b_1 > 0$. if $b_1 \geq 2$ then $4 = 3b_1 + b_2 \geq 6$, again a contradiction. Hence $b_1 = 1$. From $3b_1 + b_2 = 4$ we have $b_2 = 1$ and hence $b_3 = 1$.

Thus, $\underline{b} = (1, 1, \dots 1)^T$ as required.