Lecture 8
Algorithmic Techniques Part 3

COMP26120

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March 2019

Farmer and his Field

A farmer has an area of A square kilometers to be planted with a combination of wheat and barley. A limited amount F of fertilizer and P of insecticide can be used, each of which is required in different amounts per unit area for wheat (F_1, P_1) and barley (F_2, P_2).

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- Must fit in the field $\quad x_1 + x_2 \leq A$
- Need enough fertilizer
- Need enough pesticide

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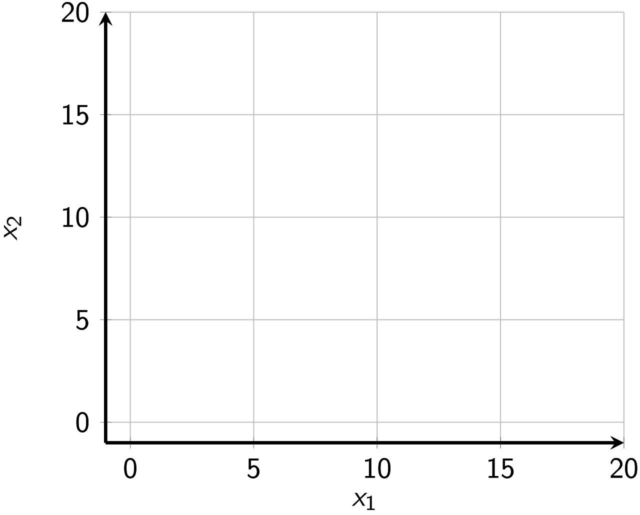
- Need enough fertilizer $F_1x_1 + F_2x_2 \leq F$
- Need enough pesticide $P_1x_1 + P_2x_2 \leq P$

Maximising

Let $A = 20$, $F = P = 50$, $F_1 = 2$, $F_2 = 3$, $P_1 = 3$, $P_2 = 2$, $S_1 = 5$, $S_2 = 4$

We get

-
-
-



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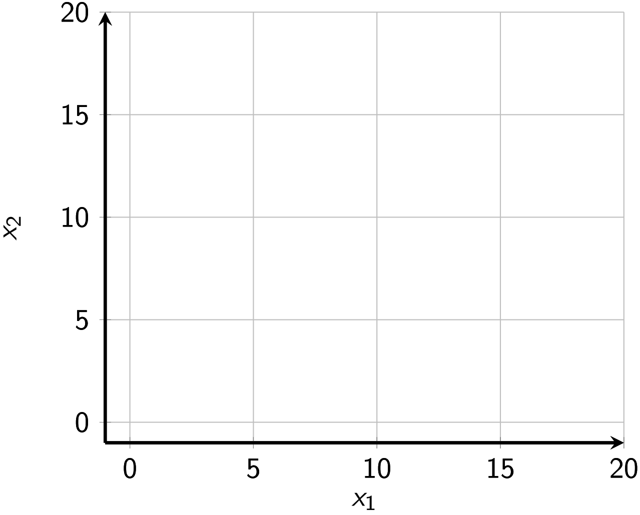

We get

- $x_1 + x_2 \leq 20$
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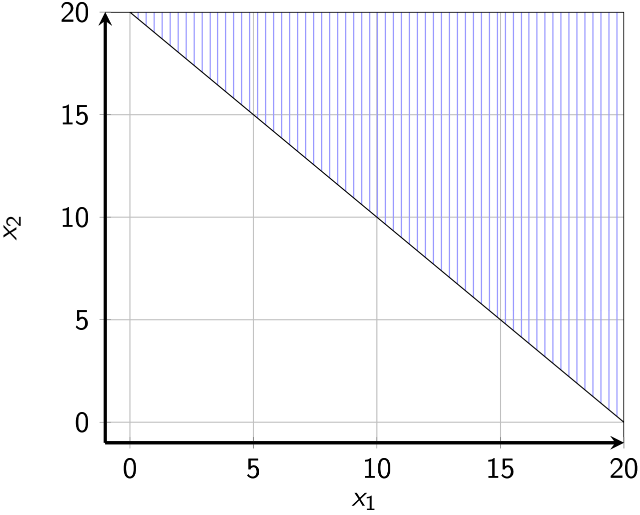

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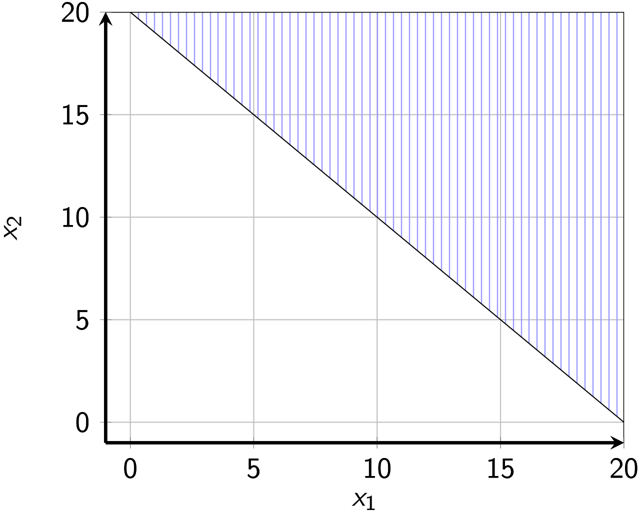

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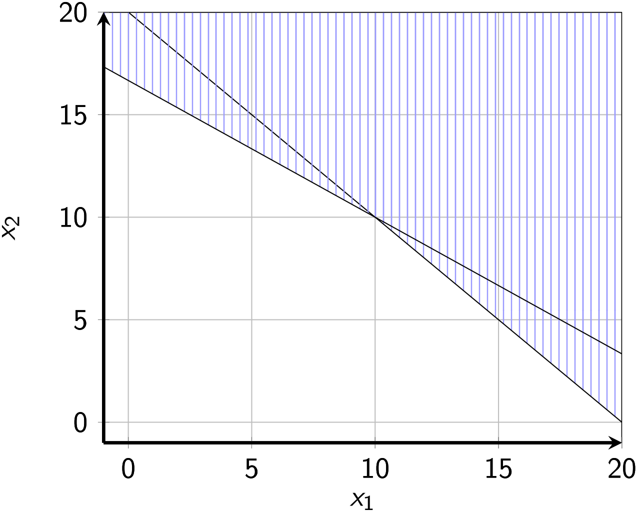

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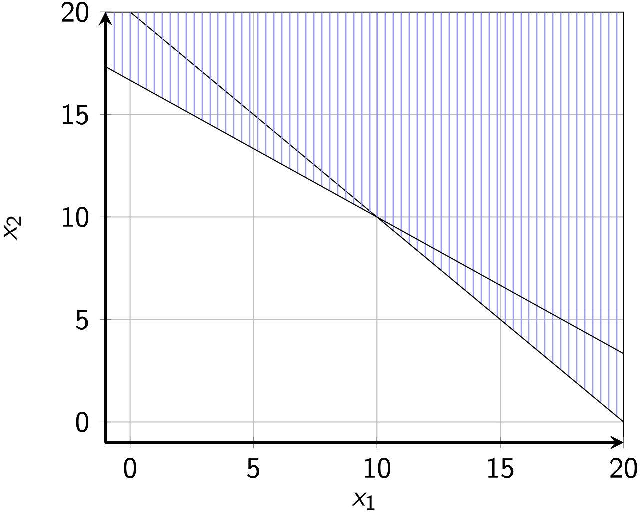

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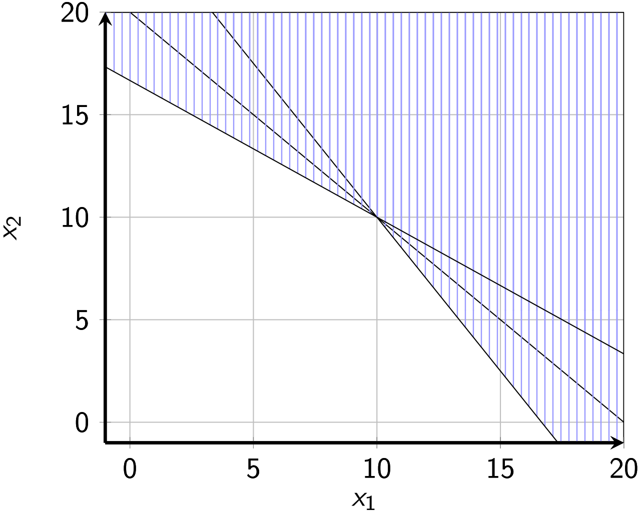

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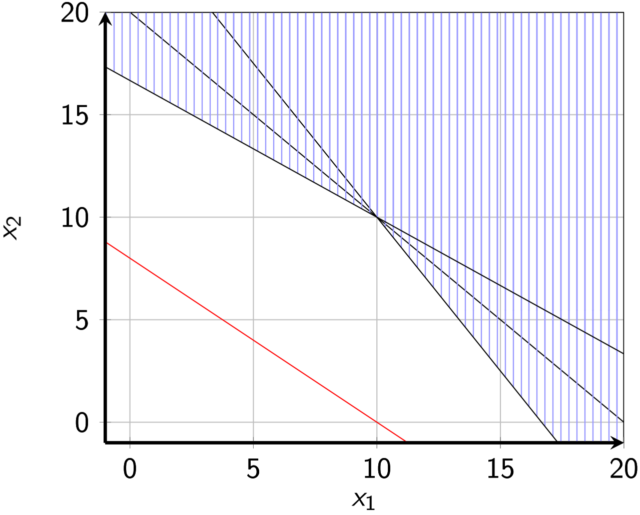

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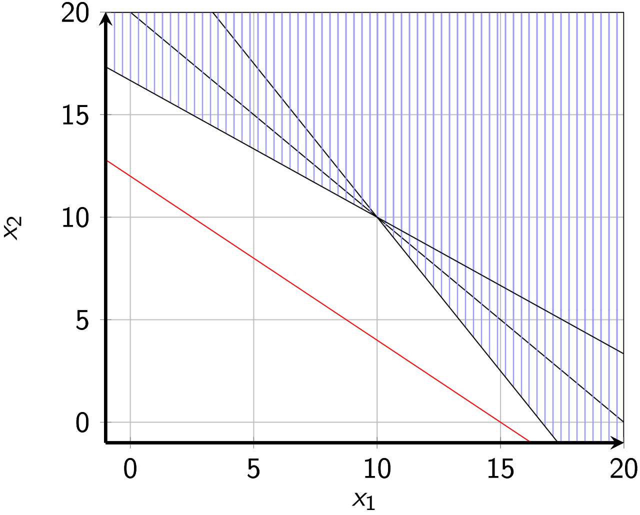

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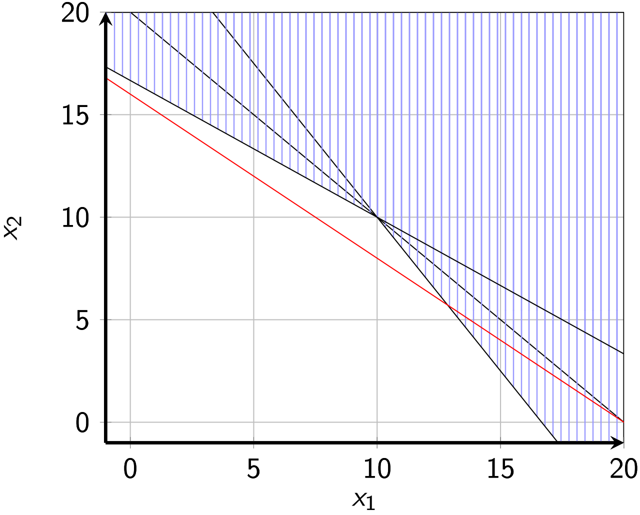

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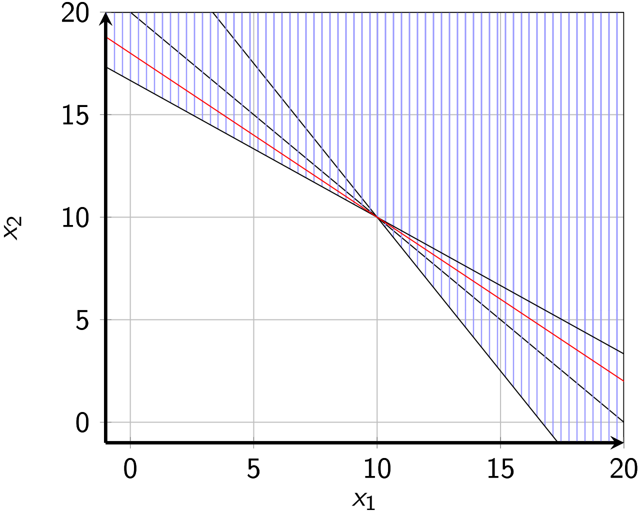

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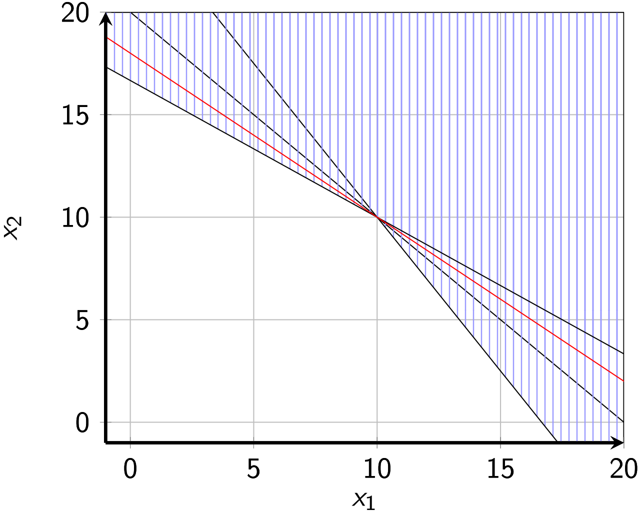

Optimisation Problem

Maximize:	$z = f(x, y) = 5x_1 + 4x_2$
Subject to:	$x_1 + x_2 \leq 20$
	$2x_1 + 3x_2 \leq 50$
	$3x_1 + 2x_2 \leq 50$
	$x_1 \geq 0, x_2 \geq 0$

Optimisation Problem

The general form is

$$\begin{array}{ll} \text{Maximize:} & \mathbf{c}^T\mathbf{x} \\ \hline \text{Subject to:} & \mathbf{A}\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \geq 0 \end{array}$$

where $\mathbf{x}$ is a vector of n variables, $\mathbf{c}$ and $\mathbf{b}$ are vectors of n and m *known* coefficients, $\mathbf{A}$ is matrix of $n \times m$ *known* coefficients

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So here

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \mathbf{c} = \begin{bmatrix} 5 \\ 4 \end{bmatrix} \quad \mathbf{b} = \begin{bmatrix} 20 \\ 50 \\ 50 \end{bmatrix} \quad \mathbf{A} = \begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 3 & 2 \end{bmatrix}$$

What does it mean?

$$\begin{array}{ll} \text{Maximize:} & \mathbf{c}^T\mathbf{x} \\ \hline \text{Subject to:} & \mathbf{A}\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \geq 0 \end{array}$$

Is asking us to find values for $\mathbf{x}$ that makes $\mathbf{c}^T\mathbf{x}$ as big as possible.

There may be many feasible solutions and many optimal solutions (we revisit this idea)

That Farmer Again

What if our Farmer wanted to use up all of her Pesticide and wants at least 5 units of wheat?

We add the constraints

- $P = P_1 + P_2$
- $x_1 \geq 5$

But these constraints don't fit into our format. But can they?

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$x_1 \geq 5$ is equivalent to $-x_1 \leq -5$

Making Meatloaf

A butcher has pork and beef available to him to make meatloafs. Each meatloaf should weight at least 1 kilo. The customers demand that the fat content of a meatloaf should be no more than 25%. The beef that the butcher works with contains 20% fat, and the pork contains 32% fat. If the beef costs 80p per kilo and the pork costs 60p, how should the butcher mix beef and pork in the meatloaf in order to *minimize* his cost?

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Use x_1 and x_2 for amount of park and beef respectively, $x_1, x_2 \geq 0$

If minimising an *at least* we could take the smallest, e.g. $x_1 + x_2 = 1$

For the fat we need $0.2x_1 + 0.32x_2 \leq 0.25$

What do we minimise?

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What do we minimise? $80x_1 + 60x_2$

Making Meatloaf

Small Problem. If we added $x_1 \geq 0.6$, e.g. because there is lots of park to use, the constraint $x_1 + x_2 = 1$ may eliminate all feasible solutions.

A better way to capture *at least* would be $x_1 + x_2 \geq 1$ with

$$0.2x_1 + 0.32x_2 \leq 0.25(x + y)$$

which can be rewritten as

$$-0.5x_1 + 0.7x_2 \leq 0$$

We need to add some flexibility (same for at most)

Duality

A useful result that

$$\begin{array}{ll} \text{Minimise:} & \mathbf{b}^T\mathbf{y} \\ \hline \text{Subject to:} & \mathbf{A}^T\mathbf{y} \geq \mathbf{c} \\ & \mathbf{y} \geq 0 \end{array}$$

is equivalent to

$$\begin{array}{ll} \text{Maximize:} & \mathbf{c}^T\mathbf{x} \\ \hline \text{Subject to:} & \mathbf{A}\mathbf{x} \leq \mathbf{b} \\ & \mathbf{x} \geq 0 \end{array}$$

Because of how we deal with $\geq$ and $\leq$ this is the same as negating all coefficients

Sometimes we want...

It's not often, but sometimes we don't want to restrict a variable x to be non-negative

We note that all numbers (positive and negative) can be represented as the difference between two non-negative things. Therefore, x gets replaced by $y - z$ for non-negative y an z.

Sometimes we might think we want strict inequalities e.g. $x < 5$ but then the region of solutions is not closed and there may be no optimum... you generally don't want this but you can add a variable ϵ to stand for the difference between x and 5 e.g. $x + \epsilon \leq 5$

Converting to Standard Form

Equality constraint $x = a$: Split into $x \geq a$ and $x \leq a$

Constraints of the form $x \geq a$: Negate the coefficients and replace $\geq$ by $\leq$

Minimisation problem : Negate the coefficients and maximise

Does not enforce $x_j \geq 0$: Replace x_j with $y_j - z_j$ and $y_j, z_j \geq 0$

Translating Your Problem to a LP problem

First, identify the variables involved - what can we vary, what does the solution space look like?

Second, identify the objective function. Hope that it's linear!

Third, identify the constraints

Fourth, translate the constraints into the nice canonical form

Have A Go

A company manufactures two products, A and B and wants to maximise its profit. The relevant production data is as follows:

- Profit per unit: £2 and £5 respectively
- Labour time per unit: 2 hours and 1 hour respectively
- Machine time per unit: 1 hour and 2 hours respectively
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- Available labour and machine time: 80 hours and 65 hours respectively
- Labour and machine overtime cost: £15 and £10 per hour, respectively

Integer or Real Solutions?

In the general linear programming problem $\mathbf{x} \in \mathbb{R}^n$

However, if we require $\mathbf{x} \in \mathbb{Z}^n$ then we have integer linear programming

We have efficient methods for solving linear programming tasks but integer linear programming is NP-complete.

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For example, we can use it to capture the 0/1 Knapsack Problem. Item i has value v_i and weight w_i.

Maximize:	$\sum_{k=1}^{N} v_k x_k$
Subject to:	$\sum_{k=1}^{N} w_k x_k \leq c$
	$x_i \geq 0$
	$x_i n \in \mathbb{N}$

3SAT Can be Reduced to Integer Linear Programming

Let P be a 3SAT problem over variables $x_1, \ldots, x_n$ such that $P_i = a_1 \vee a_2 \vee a_3$ is the ith clause where a_i is a variable of its negation.

If $a_i = x_i$ let $|a_i| = z_i$, otherwise $a_i = \neg x_i$ and $|a_i| = 1 - z_i$

Each clause $a_1 \vee a_2 \vee a_3$ gets converted to a constraint $|a_1| + |a_2| + |a_3| > 0$

We also add the constraints $z_i \in \mathbb{N}$ and $z_i \leq 1$

We don't need to maximise anything, juts find a solution.

Shortest Path as Linear Programming

Given graph (V, E, W) and source s and target t

Minimize

$$\sum_{u,v \in V} x_{u,v} W(u,v)$$

Subject to a constraint of the following form for each vertex u,

$$\sum_{v \in V} x_{u,v} - \sum_{v \in V} x_{v,u} = \begin{cases} 1 & \textit{if } u = s \\ -1 & \textit{if } u = t \\ 0 & \textit{otherwise} \end{cases}$$

Solving Linear Programming Problems

For this lecture and the next we will focus on linear programming problems (over reals).

Today I will motivate how to think about the problem geometrically and next week we will meet the Simplex algorithm *(Examinable)*.

For integer linear programming we typically use branch-and-bound or heuristic methods.

Optima at Vertices

The constraints $\mathbf{Ax} \leq \mathbf{b}$ form a feasible region and we want to find the point in that region such that the objective function evaluates to the maximal value

Due to the form of $\mathbf{Ax} \leq \mathbf{b}$, the feasible region is a convex polytope e.g. every vertex has the smallest angle inside the shape

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For any linear objective function the optima only occur at the corners (vertices) of the feasible region (never inside)

The optimum is not necessarily unique - it could lie on a line or plane (in which case the solution does not rely on at least one of the variables).

Enumeration

A solution is of the form $(v_1, \ldots, v_n)$ and a solution is feasible if it is in the feasible region.

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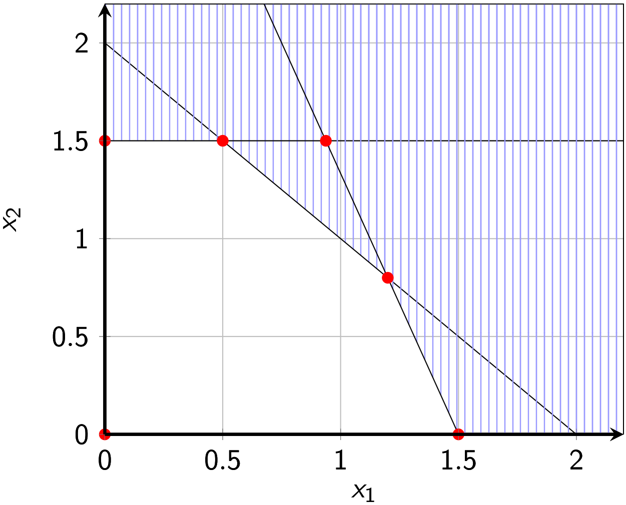
Optimum must be at an intersection of constraints, why?

A solution at an intersection is a basic solution, if it is in the feasible region it is a basic feasible solutions

Enumerative algorithm

1. Enumerate all feasible basic solutions by solving each pair of linear constraints
2. Evaluate the objective function at each solution
3. Find the biggest one

Not all basic solutions are feasible



Observations

Observation 1

The objective function defines a space of planes in an *n*-dimensional space - easier to think of as lines in a 2-dimensional space. Maximising the objective function means moving the plane in a given direction.

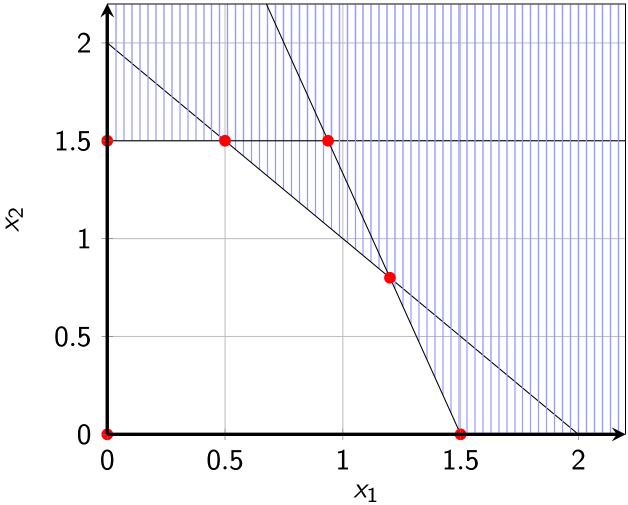
If we have one feasible basic solution we can find the current plane and rule out solutions below it and find the next solution above it. We just need a method to find this next one.

Observation 2

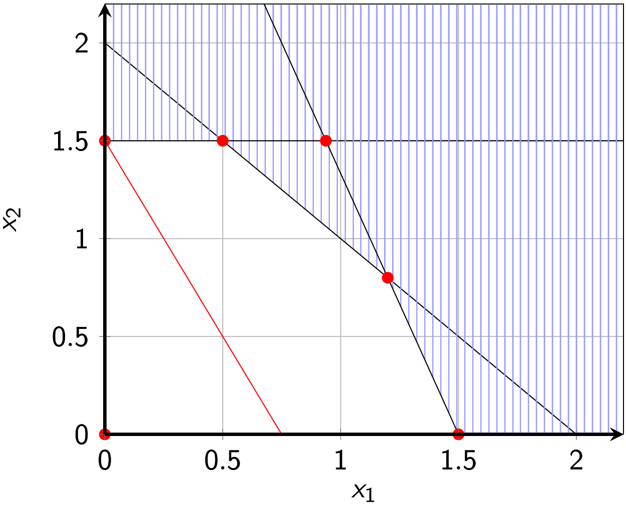
Finding all basic feasible solutions looks very similar to finding all solutions to a set of linear equations

We know how to do that (from first year)... Gaussian Elimination

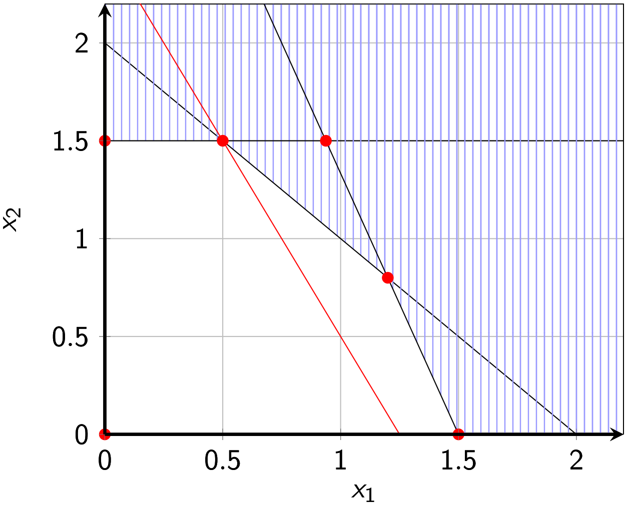
Searching vertices



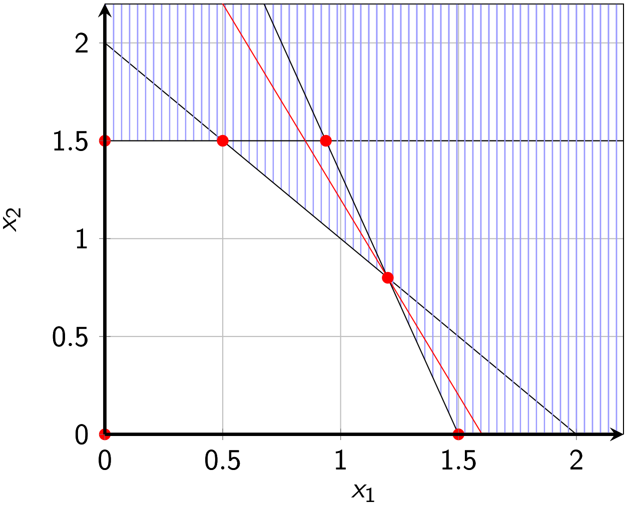
Searching vertices



Searching vertices



Searching vertices



Next time: Simplex Algorithm

The Simplex Algorithm tells us how to explore this space by representing the problem in matrix form and iteratively applying Gaussian Elimination to eliminate a variable.

The trick is knowing which variable to look at next.

We need two things before we can do this:

- Revise Gaussian Elimination
- Get our problem into a set of equations

Reminder: Gaussian Elimination

How to solve the linear equations

$$\begin{aligned} x_1 + 3x_2 + x_3 &= 9 \\ x_1 + x_2 - x_3 &= 1 \\ 3x_1 + 11x_2 + 5x_3 &= 35 \end{aligned}$$

Reminder: Gaussian Elimination

How to solve the linear equations

$$\begin{aligned} x_1 + 3x_2 + x_3 &= 9 \\ x_1 + x_2 - x_3 &= 1 \\ 3x_1 + 11x_2 + 5x_3 &= 35 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & 3 & 1 & 9 \\ 1 & 1 & -1 & 1 \\ 3 & 11 & 5 & 35 \end{array}\right]$$

Reminder: Gaussian Elimination

How to solve the linear equations

$$\begin{aligned} x_1 + 3x_2 + x_3 &= 9 \\ x_1 + x_2 - x_3 &= 1 \\ 3x_1 + 11x_2 + 5x_3 &= 35 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & 3 & 1 & 9 \\ 1 & 1 & -1 & 1 \\ 3 & 11 & 5 & 35 \end{array}\right]$$

Get into reduced row echelon form

- All non-zero rows are above any zero rows
- The leading non-zero entry is strictly to the right of the leading non-zero entry in the row above
- The leading entry in a non-zero row is 1
- A column containing a leading 1 has 0s everywhere else

By swapping rows, multiplying a row by a non-zero number, or adding a multiple of one row to another.

Slack Form

We will work with constraints in slack form

- All variables are restricted to be non-negative
- All constraints are equalities with constant, non-negative RHS

An inequality $x_1 + 2x_2 \leq 20$ has some slack e.g. some value that could be added to $x_1 + 2x_2$ whilst still satisfying the constraint. We can rewrite this as an equality $x_1 + 2x_2 + s_1 = 20$ where s_1 is a slack or basic variable

To ensure **b** is non-negative we add a surplus variable instead of a slack variable. For $-x_1 \leq -5$ we would get $-x_1 - s_1 = 5$. We could have kept our $\geq$ constraints and just used surplus variables instead of slack variables.

Finally we get

$$\begin{bmatrix} 1 & -\mathbf{c}^T & 0 \\ 0 & \mathbf{A} & \mathbf{I} \end{bmatrix} \begin{bmatrix} z \\ \mathbf{x} \\ \mathbf{s} \end{bmatrix} = \begin{bmatrix} 0 \\ \mathbf{b} \end{bmatrix}$$