Special instructions: You may work either individually or in a group of exactly two persons. Write your solutions out on paper, photocopy them and hand in both the original and the copy to SSO by 12:00 on Thursday, 23rd November, 2017. If you are working in a group, one script should be submitted for the group, with both names at the top; both members of the group will then receive the same mark. Clearly write your name(s), student ID number(s) and the words “Comp36111 Sec. B Coursework” on the front (cover) sheet and staple all sheets together. Submitted solutions must be entirely the effort of the persons whose names appear at the top of the submission.

UNIVERSITY OF MANCHESTER
SCHOOL OF COMPUTER SCIENCE

Advanced Algorithms I: Coursework for Sec. B

Time: This should take you a few hours

Please answer all questions.
Marks will be awarded for clarity and succinctness as well as correctness.

The use of electronic calculators is not recommended.
Section B

1. Let $E$ be a system of $m$ linear inequalities of the form
\[
\begin{align*}
    a_{1,1}x_1 + & \cdots + a_{1,n}x_n \leq b_1 \\
    \vdots & \vdots \\
    a_{m,1}x_1 + & \cdots + a_{m,n}x_n \leq b_m, \\
\end{align*}
\]
over the (real-valued) variables $x_1, \ldots, x_n$. Assume that the coefficients $a_{i,j}$ and $b_j$ ($1 \leq i \leq m$, $1 \leq j \leq n$) are real numbers. Consider all those inequalities in which the coefficient of $x_1$ is non-zero. Clearly these can be re-written as collections of inequalities
\[
\begin{align*}
    x_1 \leq s_i & \quad (1 \leq i \leq k) \\
    x_1 \geq t_j & \quad (1 \leq j \leq \ell),
\end{align*}
\]
for some $k, \ell \geq 0$, where $s_1, \ldots, s_k$ and $t_1, \ldots, t_\ell$ are linear expressions involving the variables $x_2, \ldots, x_n$ only.

Show how to derive a system of linear inequalities $E'$ involving only the variables $x_2, \ldots, x_n$ such that $E'$ has a (real-valued) solution if and only if $E$ has. [Hint: if $E$ has a solution, what do we know about the relations between $s_i$ and $t_j$ ($1 \leq i \leq k$, $1 \leq j \leq \ell$)?]

You must describe the construction of $E'$ precisely, and prove carefully: (i) if $E$ has a solution, then so does $E'$; and (ii) if $E'$ has a solution, then so does $E$. (4 marks)

2. Suppose now that the coefficients $a_{i,j}$ and $b_j$ in the original system $E$ are in fact integers with maximum absolute value $M$. Assume also that $E'$ is written in the general form (1), say:
\[
\begin{align*}
    d'_{1,2}x_2 + & \cdots + d'_{1,n}x_n \leq b'_1 \\
    \vdots & \vdots \\
    d'_{m,2}x_2 + & \cdots + d'_{m,n}x_n \leq b'_m, \\
\end{align*}
\]
Since the coefficients in the system $E$ are integers, then the various coefficients in the system $E'$ will be rational numbers, and so can be written in the form $a/b$, where $a$ is the numerator and $b$ the denominator. Obtain a bound on the absolute values of these numerators and denominators, as a function of $M$. (2 marks)

3. Having eliminated $x_1$, we can carry out the same process and eliminate $x_2, \ldots, x_{n-1}$. Thus, we will obtain a collection of inequalities of the form
\[
\begin{align*}
    a^*_{1,n} & \leq b^*_1 \\
    \vdots & \vdots \\
    a^*_{m,n} & \leq b^*_m,
\end{align*}
\]
where the $a^*_1, \ldots, a^*_m$ and $b^*_1, \ldots, b^*_m$ are rational numbers (i.e. of the form $a/b$). Obtain an upper bound on the absolute values of these numerators and denominators in terms of $M$ and $n$. (6 marks)
4. Give an algorithm (not complicated!) for solving a system of inequalities of the form (3), if a solution exists, and obtain a bound (as a function of the constants $a_1^*, \ldots , a_m^*$ and $b_1^*, \ldots , b_m^*$) on the size of the absolute value of the solution your algorithm finds. That is, prove a statement of the form: “If (3) has a solution, then it has a solution in which $|x_n|$ is bounded by …”. (2 marks)

5. Hence derive an analogous result for (1): i.e. give an algorithm for obtaining a solution (if one exists), together with a bound on the absolute values occurring in the solution obtained, as a function of $M$ and $n$. (4 marks)

6. Suppose now that the variables $x_1, \ldots , x_n$ are restricted to range over the natural numbers. Explain how to adapt your algorithm to incorporate this restriction, and say whether the bounds you obtained above continue to hold when solutions are constrained to be natural numbers, briefly—and informally—justifying your answer. (2 marks)