1. Let $P_1, P_2$ be languages in the respective alphabets $\Sigma_1, \Sigma_2$.
   A many-one logspace reduction from $P_1$ to $P_2$ is a function $F: \Sigma_1^* \rightarrow \Sigma_2^*$, computable in space $\log(n)$, such that $x \in P_1$ iff $F(x) \in P_2$.

2. Let $G = (V,E)$ be a graph. A 3-colouring $\Theta: G$ is a function $\Theta: V \rightarrow \{0,1,2,3\}$ st. $(u,v) \in E \Rightarrow \Theta(u) \neq \Theta(v)$. The problem 3-colourability is defined as:

   **Given:** A graph $G$
   **Return:** Yes if there exists a 3-colouring of $G$.

   This problem is NP-complete.

3. $\exists x \in \Sigma_1^* (P(x) \land P_2(x)) \quad \circled{1}$
   $\exists x \in \Sigma_1^* (P(x) \land P_1(x)) \quad \circled{2}$
   $\exists x \in \Sigma_1^* (P(x) \land P_2(x)) \quad \circled{3}$
   $\exists x \in \Sigma_1^* (P_1(x) \land P_2(x)) \quad \circled{4}$

   Each triple is obviously satisfiable.

4. We must show:
   a) If $G$ has a 3-colouring, then $\overline{\exists \Theta G}$ is satisfiable.
   b) If $\overline{\exists \Theta G}$ is satisfiable, $G$ has a 3-colouring.

   For a) pick any 3-colouring $\Theta^*_0$ of $G$ and further let $\Theta^*_1, \Theta^*_2$ be the $\exists \Theta$-3-colourings obtained by cycling the colours $0 \rightarrow 1 \rightarrow 2$ in the obvious way.
Let $A = \{ \Theta_0, \Theta_1, \Theta_2 \}$; let $p^\Theta = A$ and let $p^i_j$ be interpreted as suggested in the question.

(1)

Trivially

$$\forall \Theta \forall x \forall y \forall z \left( p^\Theta (x) \land p^\Theta (y) \right)$$

(2)

If $i \neq k$ then no $\Theta \in A$ assigns any vertex both $i$ and $k$, where

$$\forall \Theta \forall x \forall y \forall z \left( p^\Theta (x) \land p^\Theta (y) \right) \quad \text{all } i, k \neq j$$

(3)

Some $\Theta_0$ assigns some colours to vertex $i$; $\Theta_1$ and $\Theta_2$ will assign the remaining colours, where

$$\forall \Theta \forall x \forall y \forall z \left( p^i_j (x) \land p^k_l (y) \right) \quad \text{all } i, j \neq k \quad \text{all } k \neq j$$

(4)

Finally, no $\Theta \in A$ assigns the same colour to adjacent vertices, so

$$\forall \Theta \forall x \forall y \forall z \left( p^i_j (x) \land p^k_l (y) \right) \quad (i, j) \in \mathcal{E}, \text{ all } k \neq j$$

Hence

$$\forall \Theta \forall x \forall y \forall z \left( p^i_j (x) \land p^k_l (y) \right) \quad \text{all } i, j \neq k \quad \text{all } k \neq j$$

Conversely, suppose $\forall \Theta \forall x \forall y \forall z \left( p^i_j (x) \land p^k_l (y) \right)$, so that (1)-(4) hold. Let $P \subseteq A$ be the set of objects satisfying $p$. By (1) $|P| \leq 3$, and by (3) $P \neq \emptyset$. Pick $a \in P$.

For $i, 1 \leq i \leq 3$ let the moment. By (3) no element of $P$ satisfies more than one of the predicates $p^i_1, p^2_1, p^3_1$ yet by (3), some element of $P$ satisfies $p^i_1$, some element of $P$ satisfies $p^2_1$, and some element of $P$ satisfies $p^3_1$. Hence $a$ must be one of these. Thus the function

$$Q(i) = j \quad \text{where } j \text{ is the unique number s.t. } \forall \Theta \forall x \forall y \forall z \left( p^i_j (x) \land p^k_l (y) \right)$$

makes sense, and maps $V \to \{0, 1, 3, 2\}$. By (4) $Q$ is a colouring of $G$.
5. The answer to 4. shows us that the satisfiability problem for $\mathcal{N}^c$ is NP-hard.

[Added note: In fact, this problem is also in NP and hence is NP-complete. However, this is far from obvious, and certainly not established by the answer to q 4.]