COMP26120: Algorithms and Imperative Programming
Determinants and Permanents

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Outline

- In this lecture, we are going to look at methods for computing $(n \times n)$-determinants :

$$\begin{vmatrix} a_{1,1} & \dots & a_{1,n} \\ \vdots & \vdots & \vdots \\ a_{n,1} & \dots & a_{n,n} \end{vmatrix}.$$

- We will begin with a simple method (which you have probably encountered in school).
- We will then describe a more sophisticated method (which you may also have encountered in school).
- We will compare their complexity.

Outline

- A permutation is a 1–1 map of a set X onto itself.
- If X is a finite set, e.g. $[0, n-1] = \{0, \ldots, n-1\}$, this means just jumbling X up.
- For example, if $n = 4$, we have the permutation

$$0 \mapsto 3 \qquad 1 \mapsto 0 \qquad 2 \mapsto 1 \qquad 3 \mapsto 2,$$

  which we might more easily depict as

$$[0, 1, 2, 3] \mapsto [3, 0, 1, 2].$$

- If σ and τ are permutations, their product , $\sigma \cdot \tau$, is their composition:

$$\sigma \cdot \tau(n) = \tau(\sigma(n)).$$

- The number of permutations on an n-element set is

$$n! = 1 \cdot 2 \cdot \ \cdots \ \cdot (n-1) \cdot n.$$

- A simple inductive proof shows that, for all $n \geq 4$:

$$2^n \leq n! \leq 2^{n^2}.$$

- That is: the function $n \mapsto n!$ is $\Omega(2^n)$ and $O(2^{n^2})$.

- A special kind of permutation is a transposition . If $X = [0, n-1]$ the transposition $\sigma = (i, j)$ given by

$$\sigma(k) = \begin{cases} j & \text{if } k = i \\ i & \text{if } k = j \\ k & \text{ow.} \end{cases}$$

- It is easy to show that every permutation can be written as the product of transpositions. E.g.the permutation

$$[0, 1, 2, 3] \mapsto [1, 2, 3, 0].$$

can be written as

$$(0, 1) \cdot (0, 2) \cdot (0, 3) = (0, 3) \cdot (1, 3) \cdot (2, 3).$$

- Note that this is not unique! We define the parity of σ, written $\text{sgn}(\sigma)$, to be 1 if σ is the product of an even number of permutations, and -1 otherwise.

- For this to make sense, we must show that, if σ can be written as an even number of permutations, then it cannot be written as an odd number of permutations.
- To see this, first define

$$p(x_0, \ldots, x_{n-1}) = \prod_{i<j} (x_j - x_i),$$

and then write

$$s(\sigma) = \frac{p(\sigma(x_0), \ldots, \sigma(x_{n-1}))}{p(x_0, \ldots, x_{n-1})}$$

The following are easy to check:

$$\begin{aligned} s(\sigma) &= \pm 1 \\ s(\sigma \cdot \tau) &= s(\sigma) s(\tau) \\ s(t) &= -1 \qquad \text{for any transposition } t. \end{aligned}$$

Thus, $\operatorname{sgn}(\sigma) = s(\sigma)$, and so is well-defined.

Outline

- Consider a pair of simultaneous linear equations

$$\begin{aligned} a_{1,1}x_1 + a_{1,2}x_2 &= b_1 \\ a_{2,1}x_1 + a_{2,2}x_2 &= b_2, \end{aligned}$$

which we can write in matrix form as

$$\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}.$$

We can solve these by taking an appropriate $a_{2,1}/a_{1,1}$ time the first equation from the second:

$$(a_{2,2} - (a_{2,1}/a_{1,1})a_{1,2})x_2 = b_2,$$

as long as $(a_{2,2} - (a_{2,1}/a_{1,1})a_{1,2}) \neq 0$, or, equivalently,

$$a_{1,1}a_{2,2} - a_{1,2}a_{2,1} \neq 0.$$

- The quantity

$$a_{1,1}a_{2,2} - a_{1,2}a_{2,1} \neq 0.$$

is known as the determinant of the matrix

$$\begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix},$$

and us usually denoted

$$\begin{vmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{vmatrix}.$$

The original system of equations has a unique solution if and only if this determinant is non-zero.

- Similar remarks apply to the trio of simultaneous linear equations

$$\begin{aligned} a_{1,1}x_1 + a_{1,2}x_2 + a_{1,3}x_3 &= b_1 \\ a_{2,1}x_1 + a_{2,2}x_2 + a_{2,3}x_3 &= b_2 \\ a_{3,1}x_1 + a_{3,2}x_2 + a_{3,3}x_3 &= b_3, \end{aligned}$$

which we can write in matrix form as

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}.$$

These equations have a unique solution if and only if

$$a_{1,1} \cdot \begin{vmatrix} a_{2,2} & a_{2,3} \\ a_{3,2} & a_{3,3} \end{vmatrix} - a_{1,2} \cdot \begin{vmatrix} a_{2,1} & a_{2,3} \\ a_{3,1} & a_{3,3} \end{vmatrix} + a_{1,3} \cdot \begin{vmatrix} a_{2,1} & a_{2,2} \\ a_{3,1} & a_{3,2} \end{vmatrix} \neq 0.$$

- More generally, the n-tuple of simultaneous linear equations

$$\begin{pmatrix} a_{1,1} & \dots & a_{1,n} \\ \vdots & \vdots & \vdots \\ a_{n,1} & \dots & a_{n,n} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}.$$

has a unique solution if and only if the determinant ,

$$\begin{vmatrix} a_{1,1} & \dots & a_{1,n} \\ \vdots & \vdots & \vdots \\ a_{n,1} & \dots & a_{n,n} \end{vmatrix} =$$

$$a_{1,1} \begin{vmatrix} a_{2,2} & \dots & a_{2,n} \\ \vdots & \vdots & \vdots \\ a_{n,2} & \dots & a_{n,n} \end{vmatrix} - \dots \pm a_{1,n} \begin{vmatrix} a_{2,1} & \dots & a_{2,n-1} \\ \vdots & \vdots & \vdots \\ a_{n,1} & \dots & a_{n,n-1} \end{vmatrix},$$

is non-zero.

- It's worth writing this out for $n = 3$:

$$\begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3} \\ a_{2,1} & a_{2,2} & a_{2,3} \\ a_{3,1} & a_{3,2} & a_{3,3} \end{vmatrix} =$$

$$a_{1,1} \cdot \begin{vmatrix} a_{2,2} & a_{2,3} \\ a_{3,2} & a_{3,3} \end{vmatrix} - a_{1,2} \cdot \begin{vmatrix} a_{2,1} & a_{2,3} \\ a_{3,1} & a_{3,3} \end{vmatrix} + a_{1,3} \cdot \begin{vmatrix} a_{2,1} & a_{2,2} \\ a_{3,1} & a_{3,2} \end{vmatrix} =$$

$$a_{1,1} \cdot (a_{2,2}a_{3,3} - a_{2,3}a_{3,2}) - a_{1,2} \cdot (a_{2,1}a_{3,3} - a_{2,3}a_{3,1}) + a_{1,3} \cdot (a_{2,1}a_{3,2} - a_{2,2}a_{3,1}) =$$

$$a_{1,1}a_{2,2}a_{3,3} - a_{1,1}a_{2,3}a_{3,2} - a_{1,2}a_{2,1}a_{3,3} + a_{1,2}a_{2,3}a_{3,1} + a_{1,3}a_{2,1}a_{3,2} - a_{1,3}a_{2,2}a_{3,1}.$$

- Notice how, in each term, the left indices are 1, 2, 3 and the right indices are a permutation of this sequence.

- More generally, we have

$$\begin{vmatrix} a_{1,1} & \dots & a_{1,n} \\ \vdots & \vdots & \vdots \\ a_{n,1} & \dots & a_{n,n} \end{vmatrix} = \sum_{\sigma \in \text{perm}\{1,\dots,n\}} \text{sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)}.$$

- Obvious question: how do we compute this quantity?

- The definition

$$\begin{vmatrix} a_{1,1} & \dots & a_{1,n} \\ \vdots & \vdots & \vdots \\ a_{n,1} & \dots & a_{n,n} \end{vmatrix} =$$

$$a_{1,1} \begin{vmatrix} a_{2,2} & \dots & a_{2,n} \\ \vdots & \vdots & \vdots \\ a_{n,2} & \dots & a_{n,n} \end{vmatrix} - \cdots \pm \begin{vmatrix} a_{2,1} & \dots & a_{2,n-1} \\ \vdots & \vdots & \vdots \\ a_{n,1} & \dots & a_{n,n-1} \end{vmatrix}$$

  gives us an easy recursive solution.
- Alternatively, we could step through the permutations of $\{1, \dots, n\}$ and add up the terms of

$$\sum_{\sigma \in \mathrm{perm}\{1,\dots,n\}} \mathrm{sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)}.$$

- But both of these methods take exponential time.

- While we are on the subject of determinants, we contrast the determinant of the matrix

$$\begin{pmatrix} a_{1,1} & \dots & a_{1,n} \\ \vdots & \vdots & \vdots \\ a_{n,1} & \dots & a_{n,n} \end{pmatrix}$$

which we defined as

$$\sum_{\sigma \in \mathrm{perm}\{1,\dots,n\}} \mathrm{sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)},$$

with the permanent of the same matrix, defined as

$$\sum_{\sigma \in \mathrm{perm}\{1,\dots,n\}} \prod_{i=1}^{n} a_{i,\sigma(i)},$$

- Obvious question: how do we compute this quantity?
- We will see below that the answers to these questions are very different.

Outline

- An $n \times n$ matrix A is called upper triangular if all the entries below the diagonal are zero:

$$\begin{pmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n-1}, & a_{1,n} \\ 0 & a_{2,2} & \dots & a_{2,n-1} & a_{2,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & a_{n-1,n-1} & a_{n-1,n} \\ 0 & 0 & \dots & 0 & a_{n,n} \end{pmatrix}$$

- The determinant of an upper triangular matrix is simply the product of its diagonal elements:

$$\prod_{i=1}^{n} a_{i,i} = a_{1,1} \cdot a_{2,2} \cdot \ \cdots \ \cdot a_{n-1,n-1} \cdot a_{n,n}.$$

- Furthermore, if A is any $n \times n$ matrix, its determinant is unaffected by the following operations on:
  - transposing two rows;
  - transposing two columns;
  - adding a multiple of one row of to another;
  - adding a multiple of one column of to another/
- And if any row (or column) of a matrix is zero, then the determinant is zero.
- This gives us a was of computing determinants mroe efficiently:

- First, we may transpose columns (if necessary) so that $a_{1,1} \neq 0$. (If this is impossible, the determinant is 0.)

$$\begin{pmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n-1}, & a_{1,n} \\ a_{2,1} & a_{2,2} & \dots & a_{2,n-1} & a_{2,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n-1,1} & a_{n-1,2} & \dots & a_{n-1,n-1} & a_{n-1,n} \\ a_{n,1} & a_{n,2} & \dots & a_{n,n-1} & a_{n,n} \end{pmatrix}$$

- Now take suitable multiples of the first row off the subsequent rows so as to zero the first column:

$$\begin{pmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n-1}, & a_{1,n} \\ 0 & a_{2,2} & \dots & a_{2,n-1} & a_{2,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & a_{n-1,2} & \dots & a_{n-1,n-1} & a_{n-1,n} \\ 0 & a_{n,2} & \dots & a_{n,n-1} & a_{n,n} \end{pmatrix}$$

- Next, we may transpose columns (if necessary) so that $a_{2,2} \neq 0$. (If this is impossible, the determinant is 0.)

$$\begin{pmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n-1}, & a_{1,n} \\ 0 & a_{2,2} & \dots & a_{2,n-1} & a_{2,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & a_{n-1,2} & \dots & a_{n-1,n-1} & a_{n-1,n} \\ 0 & a_{n,2} & \dots & a_{n,n-1} & a_{n,n} \end{pmatrix}$$

- Now take suitable multiples of the second row off the subsequent rows so as to zero the second column:

$$\begin{pmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n-1}, & a_{1,n} \\ 0 & a_{2,2} & \dots & a_{2,n-1} & a_{2,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & a_{n-1,n-1} & a_{n-1,n} \\ 0 & 0 & \dots & a_{n,n-1} & a_{n,n} \end{pmatrix}$$

- Continue zeroing columns up to the $(n-1)$th:

$$\begin{pmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n-1}, & a_{1,n} \\ 0 & a_{2,2} & \dots & a_{2,n-1} & a_{2,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & a_{n-1,n-1} & a_{n-1,n} \\ 0 & 0 & \dots & 0 & a_{n,n} \end{pmatrix}$$

- Now compute the determinant as

$$\prod_{i=1}^{n} a_{i,i}.$$

- Warning: the $a_{i,j}$'s above are not the same as they appear in different matrices!

- Let us now calculate the complexity of this procedure:
  - To zero the i column below the diagonal, we need to do a transposition of columns $O(n)$, and, for each of $(n - i + 1) \leq n$ rows below *ith* row, subtract a multiple of the ith row from that row, which contains $n - i + 1 \leq n$ non-zero elements: so, $O(n)$ operations per row.
  - So cost of zeroing ith column below diagonal is $O(n^2)$.
  - There are $n - 1 \leq n$ columns to zero below the diagonal, so cost of converting to upper-triangular form is $O(n^3)$.
  - Cost of multiplying diagonal elements is $O(n)$.
- So cost of whole procedure is $O(n^3 + n) = O(n^3)$.
- Although this looks complicated, this is much better than the $\Omega(2^n)$ methods outlined above.

Outline

- We have seen how to compute the determinant of the matrix

$$\begin{pmatrix} a_{1,1} & \dots & a_{1,n} \\ \vdots & \vdots & \vdots \\ a_{n,1} & \dots & a_{n,n} \end{pmatrix}.$$

- But what about the permanent ? Remember, this is defined as

$$\sum_{\sigma \in \mathrm{perm}\{1,\dots,n\}} \prod_{i=1}^{n} a_{i,\sigma(i)}.$$

- It seems that there is no efficient method to calculate this quantity: the best-known algorithms run in exponential time.
- This fact (if it is a fact), has important consequences in the theory of combinatorial optimization and computational complexity.