COMP26120: Algorithms and Imperative Programming
Basic sorting algorithms

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- Reading for this lecture (Goodrich and Tamassia):
    - Secs. 8.1, 8.2, 8.3 (pp. 241–258).
    - Sec. 1.1.5 (pp. 11–16).

Outline

- Consider the problem of sorting a list of numbers (in ascending order).
- Quicksort is a sorting algorithm which works well in practice.

```
quicksort(L)
    if length of L ≤ 1
        return L
    remove the first element, x, from L
    L≤ := elements of L less than or equal to x
    L> := elements of L greater than x
    Lℓ := quicksort(L≤)
    Lr := quicksort(L>)
    return Lℓ + [x] + Lr
end
```

- The element x is sometimes referred to as the pivot.

- Example:

```
quicksort([x|L])                quicksort([2,9,1,3,4,0])
   if length of L ≤ 1              quicksort([1,0])
      return L                        quicksort([0])
   remove x from L                    quicksort([ ])
   compute L≤, L>                  quicksort([9, 3, 4])
   Lℓ := quicksort(L≤)                quicksort([3, 4])
   Lr := quicksort(L>)                   quicksort([ ])
   return Lℓ + [x] + Lr                  quicksort([4])
end                                   quicksort([ ])
```

- Example:

quicksort($[x|L]$)
  if length of $L \leq 1$
    return L
  remove x from L
  compute $L_{\leq}$, $L_{>}$
  $L_\ell :=$ quicksort($L_{\leq}$)
  $L_r :=$ quicksort($L_{>}$)
  return $L_\ell + [x] + L_r$
end

```
quicksort([2,9,1,3,4,0])
  quicksort([1,0])
    quicksort([0]) [0]
    quicksort([ ])
  quicksort([9, 3, 4])
    quicksort([3, 4])
      quicksort([ ])
      quicksort([4])
    quicksort([ ])
```

- Example:

quicksort($[x|L]$)
    if length of $L \leq 1$
        return L
    remove x from L
    compute $L_{\leq}$, $L_{>}$
    $L_\ell :=$ quicksort($L_{\leq}$)
    $L_r :=$ quicksort($L_{>}$)
    return $L_\ell + [x] + L_r$
end

```
quicksort([2,9,1,3,4,0])
  quicksort([1,0])
    quicksort([0]) [0]
    quicksort([ ]) [ ]
  quicksort([9, 3, 4])
    quicksort([3, 4])
      quicksort([ ])
      quicksort([4])
    quicksort([ ])
```

- Example:

```
quicksort([x|L])
    if length of L ≤ 1
       return L
    remove x from L
    compute L≤, L>
    Lℓ := quicksort(L≤)
    Lr := quicksort(L>)
    return Lℓ + [x] + Lr
end
```

```
quicksort([2,9,1,3,4,0])
   quicksort([1,0]) [0,1]
      quicksort([0]) [0]
      quicksort([ ]) [ ]
   quicksort([9, 3, 4])
      quicksort([3, 4])
         quicksort([ ])
         quicksort([4])
      quicksort([ ])
```

- Example:

quicksort($[x \mid L]$)	quicksort([2,9,1,3,4,0])
if length of $L \leq 1$	quicksort([1,0]) [0,1]
return L	quicksort([0]) [0]
remove x from L	quicksort([ ]) [ ]
compute $L_{\leq}$, $L_{>}$	quicksort([9, 3, 4])
$L_{\ell} :=$ quicksort($L_{\leq}$)	quicksort([3, 4])
$L_r :=$ quicksort($L_{>}$)	quicksort([ ]) [ ]
return $L_{\ell} + [x] + L_r$	quicksort([4])
end	quicksort([ ])

- Example:

quicksort($[x|L]$)
  if length of $L \leq 1$
    return L
  remove x from L
  compute $L_{\leq}$, $L_{>}$
  $L_\ell :=$ quicksort($L_{\leq}$)
  $L_r :=$ quicksort($L_{>}$)
  return $L_\ell + [x] + L_r$
end

```
quicksort([2,9,1,3,4,0])
   quicksort([1,0]) [0,1]
      quicksort([0]) [0]
      quicksort([ ]) [ ]
   quicksort([9, 3, 4])
      quicksort([3, 4])
         quicksort([ ]) [ ]
         quicksort([4]) [4]
      quicksort([ ])
```

- Example:

quicksort($[x|L]$)
  if length of $L \leq 1$
    return L
  remove x from L
  compute $L_{\leq}$, $L_{>}$
  L_{ℓ} := quicksort($L_{\leq}$)
  L_r := quicksort($L_{>}$)
  return $L_{\ell} + [x] + L_r$
end

```
quicksort([2,9,1,3,4,0])
   quicksort([1,0]) [0,1]
      quicksort([0]) [0]
      quicksort([ ]) [ ]
   quicksort([9, 3, 4])
      quicksort([3, 4]) [3, 4]
         quicksort([ ]) [ ]
         quicksort([4]) [4]
      quicksort([ ])
```

- Example:

quicksort($[x|L]$)
  if length of $L \leq 1$
    return L
  remove x from L
  compute $L_\leq$, $L_>$
  $L_\ell :=$ quicksort($L_\leq$)
  $L_r :=$ quicksort($L_>$)
  return $L_\ell + [x] + L_r$
end

```
quicksort([2,9,1,3,4,0])
  quicksort([1,0]) [0,1]
    quicksort([0]) [0]
    quicksort([ ]) [ ]
  quicksort([9, 3, 4])
    quicksort([3, 4]) [3, 4]
      quicksort([ ]) [ ]
      quicksort([4]) [4]
    quicksort([ ]) [ ]
```

- Example:

```
quicksort([x|L])
    if length of L ≤ 1
        return L
    remove x from L
    compute L≤, L>
    Lℓ := quicksort(L≤)
    Lr := quicksort(L>)
    return Lℓ + [x] + Lr
end
```

```
quicksort([2,9,1,3,4,0])
    quicksort([1,0]) [0,1]
        quicksort([0]) [0]
        quicksort([ ]) [ ]
    quicksort([9, 3, 4]) [3, 4, 9]
        quicksort([3, 4]) [3, 4]
            quicksort([ ]) [ ]
            quicksort([4]) [4]
        quicksort([ ]) [ ]
```

- Example:

quicksort($[x|L]$)
    if length of $L \leq 1$
        return L
    remove x from L
    compute $L_{\leq}$, $L_{>}$
    $L_\ell :=$ quicksort($L_{\leq}$)
    $L_r :=$ quicksort($L_{>}$)
    return $L_\ell + [x] + L_r$
end

quicksort([2,9,1,3,4,0]) [0,1,2,3,4,9]
    quicksort([1,0]) [0,1]
        quicksort([0]) [0]
        quicksort([ ]) [ ]
    quicksort([9, 3, 4]) [3, 4, 9]
        quicksort([3, 4]) [3, 4]
            quicksort([ ]) [ ]
            quicksort([4]) [4]
        quicksort([ ]) [ ]

- Let's see how much work is done:
- The worst case occurs when, for each recursive call, one of $L_{\leq}$ or $L_{>}$ is empty.
- Here n recursive calls are made (ignoring calls with [ ]), with the argument one element shorter each time.
- Before each recursive call, $L_{\leq}$ and $L_{>}$ must be calculated, requiring $O(|L|)$ steps.
- So if $|L| = n$, total work is order

$$n + n - 1 + \cdots + 1 = \frac{1}{2}n(n+1)$$

i.e. $O(n^2)$ (because $O(\frac{1}{2}n(n+1)) = O(n^2)$).

Outline

- Here is an algorithm with lower complexity.
- First, consider the problem of merging two sorted list to form a third sorted list.

  merge([1, 3, 5], [0, 2, 4, 6, 7]) $\Rightarrow$
  [0, 1, 2, 3, 4, 5, 6, 7]
- This algorithm will work.

merge(L_1, L_2)
  if $L_1 = [\,]$
    return L_2
  if $L_2 = [\,]$
    return L_1
  x_i =first element of L_i $(i = 1, 2)$
  $L_i' = L_i$ minus first element $(i = 1, 2)$
  If if $x_1, \leq x_2$
    return $[x_1]$+merge(L_1', L_2)
  return $[x_2]$+merge(L_1, L_2')
end

- When $\mathtt{merge}(L_1, L_2)$ is called, at most one recursive call is made, in which $|L_1| + |L_2|$ decreases by 1.
- Therefore, at most $O(n)$ recursive calls are made, where $n = |L_1| + |L_2|$ is the length of the input.
- A constant number of operations is executed for each recursive call.
- Therefore, at it takes most $O(n)$ time to run.

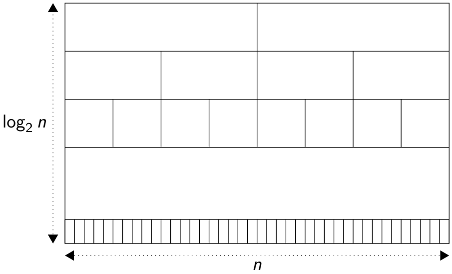
- We can now present our sorting algorithm

```
mergeSort(L)
    if |L| ≤ 1
        return L
    Split L into two roughly equal halves L = L_ℓ + L_r
    return merge(mergeSort(L_ℓ),mergeSort(L_r))
end
```

- This algorithm clearly returns a sorted list with exactly the original elements.

- How many times is the algorithm called recursively?
- The following analysis gives a rather disappointing bound:
    - Each recursive call gives rise to two others at at one greater depth of recursion.
    - Thus, each depth of iteration, there are twice as many recursive calls.
    - The maximum depth of recursion is $\lceil \log_2 n \rceil$.
    - Therefore, the number of calls is $2^{(\lceil \log_2 n \rceil)} \leq 2n$.
    - (Actually, a better bound is $n - 1$: can you see why?)
- The time taken to merge is at most $O(n)$, so this suggests (*prima facie*) a complexity bound of $O(n^2)$.

- But in fact it's not that bad.



- The total lengths of lists processed at each level of recursion is constant at $|L| = n$.
- And the total amount of work done for each call is linear in the lengths of the arguments.
- The number of times L can be halved is $O(\log n)$.
- Hence, the time complexity of `mergeSort` is $O(n \log n)$.

- Or do some algebra. Let the time taken my `mergeSort` on any list of length n be bounded be (worst case), $t(n)$. Then, ignoring constant factors

$$t(n) = 2t\left(\frac{n}{2}\right) + n$$

and, without loss of generality, we may as well assume that $t(2) \leq 2$.

- A simple induction shows that

$$t(n) \leq n \log_2 n.$$

For, $n > 2$ (and cheating quite a lot), we have

$$\begin{aligned} t(n) &= 2t\left(\frac{n}{2}\right) + n \\ &\leq 2\frac{n}{2}\log_2\left(\frac{n}{2}\right) + n \qquad \text{(ind. hyp.)} \\ &= n \log_2 n. \end{aligned}$$

Outline

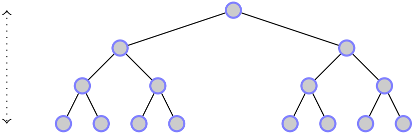
- Can we do any better than $O(n \log_2 n)$?
- In the study of algorithms, lower complexity bounds are in general extraordinarily hard to obtain.
- In the case of sorting, however, we have a qualified lower bound:

  Any algorithm which sorts a list using only number-comparison operations requires time at least $n \log_2 n$ to run.

- Let us see why this is so.

- First some basic facts about trees.
- Suppose we have a full binary tree of depth d with n vertices in total, of which ℓ are leaves.

$d = 3$



In this example, $d = 3$, $n = 15$ and $\ell = 8$.

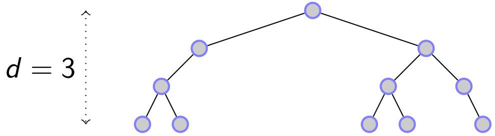
- Starting with the root at level 0, the number of vertices on each level k is 2^k. Hence

$$\ell = 2^d \qquad\qquad n = \sum_{k=0}^{d} 2^k = 2^{d+1} - 1.$$

- Otherwise expressed:

$$d = \log_2 \ell \qquad\qquad d = \log_2(n+1) - 1.$$

- If the tree is binary branching, but not full, then these equalities are replaced by inequalities.



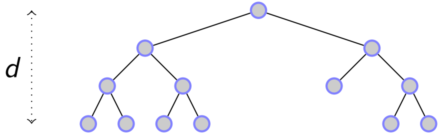
- Here we are missing some vertices. Hence:

$$\ell \leq 2^d \qquad\qquad n =\leq 2^{d+1} - 1.$$

- Otherwise expressed:

$$d \geq \log_2 \ell \qquad\qquad d \geq \log_2(n+1) - 1.$$

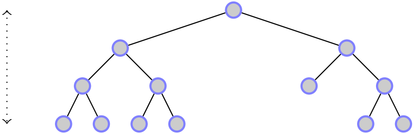
- Now suppose we have an algorithm which sorts a list by making comparisons and branching as a result of that comparison.
- The possible runs of that algorithm may be arranged as a binary tree.



- Assume without loss of generality, that the input of length n are the integers 1–n in some order $\pi(1), \ldots, \pi(n)$, where π is a permutation.
- The algorithm will then apply the inverse permutation π^{-1} to sort the list.

- There are $n!$ permutations of the numbers 1–n, each requiring a different output, and hence $n!$ leaves in the computation tree.

$t(n) = d$



- The maximum running time, $t(n)$, on inputs of length n is the (maximum) depth of the tree.
- From our inequality $d \geq \log_2(\ell)$ we obtain, assuming n even:

$$
\begin{aligned}
t(n) \geq \log_2(n!) \geq \log_2\left(\left(\frac{n}{2}\right)^{\frac{n}{2}}\right) \\
= \frac{n}{2}\log_2\left(\frac{n}{2}\right) = \frac{1}{2}n(\log_2(n) - 1).
\end{aligned}
$$

- The following is a very handy way of talking about lower bounds.
- If $f : \mathbb{N} \to \mathbb{N}$ is a function, then $\Omega(f)$ denotes the set of functions:

$$\{g : \mathbb{N} \to \mathbb{N} \mid \exists n_0 \in \mathbb{N} \text{ and } c \in \mathbb{R}^+ \text{ s.t. } \forall n > n_0,\ g(n) \geq c \cdot f(n)\}.$$

- Thus, $\Omega(f)$ denotes a *set* of functions, intuitively, the functions that grow essentially at least as fast as f.

- Notice that for $n \geq 4$, $\frac{1}{2} n \log_2(n) \geq n$.
- In particular, for sufficiently large n,

$$\frac{1}{2} n(\log_2(n) - 1) \geq \frac{1}{4} n \log_2(n).$$

- That is,

$$\frac{1}{2} n(\log_2(n) - 1) \in \Omega(n \log_2(n))$$

- Thus, we are guaranteed that any sorting algorithm based on comparisons has running time (in) $\Omega(n \log_2(n))$.
- Warning, this doesn't provide a guarantee of the complexity of *any* algorithm whatsoever. On the other hand, no one has done any better so far ...